Dependence Logic
Exercise 1

1. Let $M = \{0, 1, 2\}$ and $\tau = \{R\}$, where R is a binary ($ar(R) = 2$) relation symbol. Give an example of two non-isomorphic structures $\mathcal{M}$ and $\mathcal{M}'$ such that $dom(\mathcal{M}) = dom(\mathcal{M}') = M$ and $|R^{\mathcal{M}}| = |R^{\mathcal{M}'}|$.

2. Let f be a unary ($ar(f) = 1$) function symbol. Construct a $\{f\}$-sentence φ of first-order logic such that the following holds for all $\mathcal{M}$ of vocabulary $\{f\}$:

$$\mathcal{M} \models \varphi \Rightarrow \text{Dom}(\mathcal{M}) \text{ is infinite.}$$

3. Let $\{E\}$ be a binary relation symbol. Construct a first-order sentence φ such that

$$\mathcal{M} \models \varphi \Leftrightarrow E^{\mathcal{M}} \text{ is an equivalence relation,}$$

that is, symmetric, reflexive, and transitive.

4. For a first-order formula ϕ, consider the operations $\phi \mapsto \phi^p$ and $\phi \mapsto \phi^d$ that transform ϕ to a formula in negation normal form. Show that

$$(\phi^d)^d = (\phi^p)^p = \phi^p,$$

and that ϕ^p is logically equivalent to ϕ and ϕ^d to $\neg\phi$.

5. Let $M = \{0, 1, 2\}$. Consider the following team X of M with domain $\{x_0, x_1, x_2\}$:

	x_0	x_1	x_2
s_0	1	2	2
s_1	2	1	2
s_2	0	1	2

Does $M \models_X \phi$ hold when φ is defined as follows:

1. $\phi := x_0 = x_2$ or $\phi := \neg x_0 = x_2$
2. $\phi := \exists x_0(x_0 = x_2)$
3. $\phi := \forall x_3 =\!(x_2)$
4. $\phi := (=\!(x_0, x_1) \vee =\!(x_1, x_2))$

6. Let M and X be as above. Does $M \models_X \phi$ hold when φ is defined as follows:

1. $\phi := x_1 \subseteq x_2$
2. $\phi := x_1 \perp x_2$
3. $\phi := x_1 \perp_{x_1} x_2$
4. $\phi := (x_1, x_1)|(x_2, x_2)$