Topology II

Exercise 2 (21.9.2017)

1. Suppose, that X is a set and $(\mathcal{T}_j)_{j\in J}$ is a family of topologies in X. Prove, that

$$\mathcal{T} = \bigcap\{\mathcal{T}_j \mid j \in J\}$$

also is a topology in X.

2. Prove, that the corresponding result for the union of topologies is not true.

3. Suppose, that X is a topological space and $A \subset X$. Prove, that the following conditions are equivalent:
(1) A is dense in X.
(2) A meets every non-empty open subset of X.
(3) $\operatorname{int}(X \setminus A) = \emptyset$.

4. Suppose, that d and e are metrics in a set X, and $d(x,y) \le e(x,y)$ for all $x, y \in X$. Prove, that $\mathcal{T}_d \subset \mathcal{T}_e$.

5. Let $X = \mathbb{R}$ and $\mathcal{T}$ is the collection, which consists of $\emptyset$, $\mathbb{R}$, and all intervals of the form $]a, \infty[$, where $a \in \mathbb{R}$. Prove, that this collection is a topology in $\mathbb{R}$. What is the closure of the set $\{0\}$ in this topology?

6. Suppose, that $A_1, A_2, ...$ is a sequence of subsets of a space X. The (topological) lim sup of this sequence is the set of all elements $x \in X$ which have the property: every neighbourhood of x meets A_j for infinitely many indices $j \in \mathbb{N}$. The lim inf of this sequence is the set of all elements $x \in X$ which have the property: every neighbourhood of x meets A_j for all but a finite number of indices $j \in \mathbb{N}$ (that is, for all j bigger than some j_0).

a) Prove, that these sets are closed.

b) Prove, that these sets don't change, if we replace the sets A_j with their closures.

c) Give an example, where

$$\text{lim inf}_{j\to\infty} = \emptyset \neq \text{lim sup}_{j\to\infty} A_j.$$