Topology II

Exercise 3 (28.9.2017)

1. Prove, that the interval $[a, b[$ is an open and closed set in the space $(\mathbb{R}, \mathcal{T}_{pa})$. Determine the closure and boundary of the interval $]0, 1[$.

2. Prove, that the collection $\mathcal{B} = \{[a, b] \mid a \in \mathbb{Q},\ b \in \mathbb{R} \setminus \mathbb{Q},\ a < b\}$ is a basis for some topology in $\mathbb{R}$. Prove:
(a) $\mathcal{T}_{tav} \subset \mathcal{T}$,
(b) $\mathcal{T} \not\subset \mathcal{T}_{pa}$,
(c) $\mathcal{T}_{pa} \not\subset \mathcal{T}$.

3. What is the topology in $\mathbb{R}^3$ generated by the collection of all 2-dimensional planes?

4. Let X be a space, $A \subset X$ and $f\colon X \to \mathbb{R}$ the characteristic function of the set A, that is, $f(x) = 1$, when $x \in A$, and $f(x) = 0$, when $x \notin A$. Prove, that

$$\partial A = \{x \in X \mid f \text{ is discontinuous at the point } x\}.$$

5. Prove, that the projection $\mathrm{pr}_1\colon \mathbb{R}^2 \to \mathbb{R}$ is an open but not a closed map.

6. Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be a continuous map, such that $|f(x)| \to \infty$, when $|x| \to \infty$. Prove, that f is a closed map. Using this, prove, that every polynomial $f\colon \mathbb{R} \to \mathbb{R}$ is a closed map. [Hint: You need some results on compactness]