Solution to exercise 1.26 iv)

The final size s is the solution of equation (1.22) of the book,

$$\ln s = \frac{R_0}{1-f} \ln(f + (1-f)s) \tag{1}$$

for $R_0 > 1$. Assume throughout $s, f \in (0,1)$. Implicit differentiation with respect to f yields

$$\frac{1}{s}\frac{ds}{df} = \frac{R_0}{(1-f)^2} \ln(f + (1-f)s) + \frac{R_0}{(1-f)(f+(1-f)s)} \left(1 - s + (1-f)\frac{ds}{df}\right) \tag{2}$$

which is readily rearranged into

$$\left[\frac{1}{s} - \frac{R_0}{f + (1-f)s}\right] \frac{ds}{df} = \frac{R_0}{(1-f)^2} \left[\ln(f + (1-f)s) - \left(1 - \frac{1}{f + (1-f)s}\right)\right] \tag{3}$$

The right hand side of this equation is positive since $\ln x - (1 - 1/x)$ is positive for $x \in (0,1)$. $\frac{ds}{df}$ is therefore positive if and only if the bracketed factor on the left hand side is positive, i.e., if and only if

$$f + (1-f)s - R_0 s > 0 \tag{4}$$

Express R_0 from (1) and substitute into (4) to obtain the necessary and sufficient condition for $\frac{ds}{df} > 0$ as

$$g(s,f) := [f + (1-f)s] \ln(f + (1-f)s) - (1-f)s \ln s < 0 \tag{5}$$

Notice that $g(s,0) = g(s,1) = 0$ for all $s > 0$ and $g(1,f) = 0$ for all f. Since

$$\frac{\partial g}{\partial s} = (1-f)\left[\ln(f + (1-f)s) - \ln s\right] \tag{6}$$

and, from (1), $\ln s < 0$ is greater in absolute value than $\ln(f + (1-f)s)$, g is an increasing function of s for all $0 < f < 1$, which, together with $g(1,f) = 0$ for all f, implies (5). Alternatively, since

$$\frac{\partial^2 g}{\partial f^2} = \frac{(1-s)^2}{f + (1-f)s} > 0, \tag{7}$$

g is a convex function of f, which, together with $g(s,0) = g(s,1) = 0$, implies (5).