Dependence Logic

Exercise 4

1. Demonstrate the failure of the equivalence

$$\forall x(\psi \vee \phi) \equiv \forall x\psi \vee \phi$$

for inclusion logic formulas by providing the missing details to Example 30.

2. Let R be a binary relation symbol, $\mathcal{M}$ a finite τ-structure, and $\phi := \exists x \exists y(y \subseteq x \wedge R(x, y))$. When does $\mathcal{M} \models \phi$ hold?

3. Show the logical consequence

$$\phi \Rightarrow =((t_1, ..., t_n), t)$$

where $\phi := \forall z(z = t' \vee (t_1, \ldots, t_n, z)|(t_1, \ldots, t_n, t))$ (see the proof of Theorem 34).

4. Let $\phi \in \mathrm{FO}(=(...))[\tau]$ with free variables $x_1, \ldots, x_k$. Let R be a k-ary relation symbol not in τ. Construct a $\mathrm{FO}(=(...))[\tau \cup \{R\}]$-sentence ψ such that for all τ-structures $\mathcal{M}$ and teams X of $\mathrm{Dom}(M)$ with domain $\{x_1, \ldots, x_k\}$:

$$\mathcal{M} \models_X \phi \Leftrightarrow \mathcal{M}' \models \psi,$$

where $\mathcal{M}'$ is the extension of $\mathcal{M}$ to the vocabulary $\tau \cup \{R\}$ such that $R^{\mathcal{M}'} = \{(s(x_1), .., s(x_k)) \in \mathrm{Dom}(\mathcal{M})^k \mid s \in X\}$.

5. Let $\phi \in \mathrm{FO}(=(...))$ be a sentence of the form $\exists x_1 \ldots \exists x_n \psi$, where ψ is a quantifier-free formula. Show that $\phi \equiv \phi^*$, where ϕ^* is the first-order sentence obtained by replacing all dependence atoms in ϕ by $\top$ ($\top$ is some fixed valid formula).