Derivation of the final size equation $s(\infty) = e^{-R_0(1-s(\infty))}$
This derivation replaces p. 16 of the book.

Let us say that a contact is 'infectious' if the pathogen can make its way from the infected to the host whom the infected contacts, whereupon a new infection occurs if the contacted host was susceptible. (If the contact rate is c and the probability of transmission is p, then the rate of infectious contacts is pc.)

Suppose first that no one dies, so that the total number of hosts N is constant. Recall that $\Lambda(t)$, the force of infection, is the rate at which a susceptible 'decays' into an infected, i.e., we have

$$\frac{dS(t)}{dt} = -\Lambda(t)S(t) \quad (1)$$

or, dividing with N and defining $s(t) = S(t)/N$, we have

$$\frac{ds(t)}{dt} = -\Lambda(t)s(t) \quad (2)$$

$s(t)$ is the fraction of susceptibles at time t, with initial condition $s(0) = 1$. Integrating (2) yields

$$s(\infty) = e^{-\int_0^\infty \Lambda(t)dt} \quad (3)$$

Now we determine the exponent in (3). Since $\Lambda(t)$ is the rate of getting infected, $\Lambda(t)dt$ is the probability that a host receives an infectious contact in dt. Since dt is infinitesimal, there is either no contact (with probability $1 - \Lambda(t)dt$) or one contact (with probability $\Lambda(t)dt$) but no more[1]. Hence the average *number* of infectious contacts received per host in dt is $(1 - \Lambda(t)dt) \cdot 0 + \Lambda(t)dt \cdot 1 = \Lambda(t)dt$ (same as the probability; cf. Bernoulli distribution). The number of infectious contacts made in dt in the whole population (to anybody) is $\Lambda(t)Ndt$, and the total number of infectious contacts made during the entire course of the epidemic is

$$N \int_0^\infty \Lambda(t)dt \quad (4)$$

[1]More precisely, two or more contacts occur with probability $O(dt^2)$, which is negligible.

Recall that an 'infectious contact' makes a new infection only if the receiver is susceptible, so (4) is not the number of infections, but the number of contacts that could make an infection.

There is another way of calculating the total number of infectious contacts: During the entire course of the epidemic, a total of $(1-s(\infty))N$ hosts get infected, and each one of these makes on average R_0 infectious contacts before recovery. [In the beginning (when ca all contacts are made to susceptibles) each infected makes R_0 new infections; but also later they make the same number of infectious contacts, only some of the contacts go to already infected hosts and therefore do not make new infections.] Hence the total number of infectious contacts is $(1-s(\infty))NR_0$, and comparing to (4), we have

$$\int_0^\infty \Lambda(t)dt = (1-s(\infty))R_0 \tag{5}$$

Substituting into (3) yields the final size equation

$$s(\infty) = e^{-R_0(1-s(\infty))} \tag{6}$$

By now it should also be clear why $\int_0^\infty \Lambda(t)dt$ is called the *cumulative force of infection*: it is the expected number of infectious contacts one host receives during the entire epidemic[2].

Note that we did not make particular assumptions other than well-mixing and constant population size (e.g. it is irrelevant whether we take the SIR model as in exercise 1.22 of the book or we assume that recovery occurs after a fixed time ΔT as on p. 16).

If some infecteds die, and therefore the number of hosts $N(t)$ is not constant, then we cannot make the step from (1) to (2). To help this, let N_0

[2]The actual number of infectious contacts received by a focal host individual is of course random, and it follows a Poisson distribution with expectation $\lambda = \int_0^\infty \Lambda(t)dt$. This is because there is a very large number of infectious contacts ($N\int_0^\infty \Lambda(t)dt, N \to \infty$) that the host could in principle receive during the epidemic, but each such contact comes to him with only a very small probability ($1/N$). The actual number of contacts is binomially distributed but with these extreme parameters; and the limit of the binomial distribution with large number of trials, small probability of success and finite expectation is the Poisson distribution. The probability of remaining susceptible ($s(\infty)$) is the Poisson probability of getting zero infectious contacts, $P(0) = e^{-\lambda} = e^{-\int_0^\infty \Lambda(t)dt}$.

denote the initial size of the host population ($N(0) = S(0) = N_0$), and let $s(t) = S(t)/N_0$, $n(t) = N(t)/N_0$. We can divide (1) with N_0 and arrive at (2) for the (new) definition $s(t) = S(t)/N_0$. There are however two places where the above derivation is in real trouble. First, the total number of infectious contacts (the analogue of (4)) is $\int_0^\infty N(t)\Lambda(t)dt$, and we could not obtain the cumulative force of infection for the left hand side of (5). Second, if the contact rate decreases as $N(t)$ decreases, then those who get infected late in the epidemic cannot make as many contacts as those who got infected early; and therefore it is no longer true that each infected makes R_0 infectious contacts. These two problems mean that in general the final size is *not* given by (6) if the number of hosts decreases due to death.

In the special case of mass action, however, the two problems cancel each other. To see this, the trick is to place the $I(t)$ infecteds who are present in the real population at time t into an imaginary population with population size N_0. With mass action, the contact rate is proportional to population size. In the real epidemic, the $I(t)$ infecteds make a total of $N(t)\Lambda(t)dt$ infectious contacts in dt; but in a population of N_0, the same $I(t)$ infecteds would make proportionally more, $N_0\Lambda(t)dt$ contacts. Integrating over the whole epidemic, there would be $N_0 \int_0^\infty \Lambda(t)dt$ infectious contacts in the imaginary population having the real number of infecteds at all times. The real number of infecteds during the whole epidemic is $(1 - s(\infty))N_0$. In the imaginary population, each of these would make R_0 infectious contacts. Hence we obtain $N_0 \int_0^\infty \Lambda(t)dt = (1 - s(\infty))N_0 R_0$, which is equation (5), and (6) follows as before.