For homework set 4: Fodor's lemma implies the Delta-lemma

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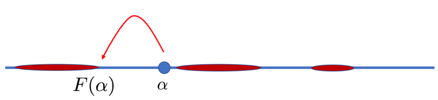
1 The finite case

CLAIM: Let W be a set of finite sets such that $|W| = \aleph_1$. Then there is $W' \subseteq W$ such that $|W'| = \aleph_1$ and W' is a Δ-system i.e. there is a "root" r such that for any two different $x, y \in W'$ we have $x \cap y = r$.

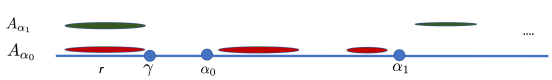
Note: The x and y really have to be *different* elements of W'. It is perfectly possible that $r = \emptyset$. Then it just means that W' is a disjoint family. Anyway, r is finite.

Since $|\bigcup W| = \aleph_1$, we may assume w.l.o.g. that $\bigcup W = \aleph_1$. Let $W = \{A_\alpha : \alpha < \omega_1\}$ (without repetition). For $\alpha < \omega_1$ limit let

$$F(\alpha) = sup(\alpha \cap A_\alpha).$$



Since $S = \{\alpha < \omega_1 : \alpha \text{ limit}\}$ is stationary, Fodor gives a stationary $S' \subseteq S$ such that F restricted to S' has a constant value, say γ. Since there are only countably many finite subsets of γ and $\gamma \cap A_\alpha$ is always finite, there is, by the Pigeon Hole Principle, $S'' \subseteq S'$ of cardinality $\aleph_1$ and a fixed set r such that $\gamma \cap A_\alpha = r$ for all $\alpha \in S''$. Let $\alpha_0 = min(S'')$. Choose $\alpha_1 \in S''$ such that $\alpha_1 > max(A_{\alpha_0})$. Then $\alpha_1 \cap A_{\alpha_1} = \alpha_0 \cap A_{\alpha_0} = r$ and hence $A_{\alpha_0} \cap A_{\alpha_1} = r$. Now we can keep going and we get a sequence $\alpha_\xi, \xi < \omega_1$, such that $A_{\alpha_\xi} \cap A_{\alpha_\zeta} = r$ for all $\xi < \zeta < \omega_1$. The set $W' = \{A_\xi : \xi < \omega_1\}$ is the set we desired.



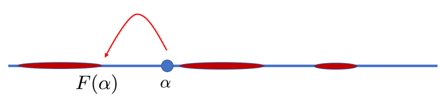
2 The countable case

CLAIM: Assume CH. Let W be a set of countable sets such that $|W| = \aleph_2$. Then there is $W' \subseteq W$ such that $|W'| = \aleph_2$ and W' is a Δ-system i.e. there is a "root" r such that for any two different $x, y \in W'$ we have $x \cap y = r$.

Note: The x and y really have to be *different* elements of W'. It is perfectly possible that $r = \emptyset$. Then it just means that W' is a disjoint family. Anyway, r is countable.

Since $|\bigcup W| = \aleph_2$, we may assume w.l.o.g. that $\bigcup W = \aleph_2$. Let $W = \{A_\alpha : \alpha < \omega_2\}$ (without repetition). For $\alpha < \omega_2$ of cofinality ω_1 let

$$F(\alpha) = sup(\alpha \cap A_\alpha).$$



Since $S = \{\alpha < \omega_2 : cof(\alpha) = \omega_1\}$ is stationary, Fodor gives a stationary $S' \subseteq S$ such that F restricted to S' has a constant value, say γ. Since we assume CH, there are only $\aleph_1$ countable subsets of γ. Since $\gamma \cap A_\alpha$ is always a countable subset of γ, one of the sets $\gamma \cap A_\alpha$ has to occur $\aleph_2$ times. Hence there is, by the Pigeon Hole Principle, a set $S'' \subseteq S'$ of cardinality $\aleph_2$ and a fixed set r such that $\gamma \cap A_\alpha = r$ for all $\alpha \in S''$. Let $\alpha_0 = min(S'')$. Choose $\alpha_1 \in S''$ such that $\alpha_1 > sup(A_{\alpha_0})$. Then $\alpha_1 \cap A_{\alpha_1} = \alpha_0 \cap A_{\alpha_0} = r$ and hence $A_{\alpha_0} \cap A_{\alpha_1} = r$. Now we can keep going and we get a sequence $\alpha_\xi, \xi < \omega_2$, such that $A_{\alpha_\xi} \cap A_{\alpha_\zeta} = r$ for all $\xi < \zeta < \omega_2$. The set $W' = \{A_\xi : \xi < \omega_2\}$ is the set we desired.