Complexity theory
Department of Mathematics and Statistics
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Exercise set 6

Read chapters 2.1–2.3 of the book.

Exercise 1. Prove that allowing the certificate to be of size *at most* $p(|x|)$ rather than equal to $p(|x|)$ in the definition of **NP** makes no difference. That is, show that for every polynomial-time Turing machine M and polynomial $p : \mathbb{N} \to \mathbb{N}$, the language

$$\{x \in \{0,1\}^* : \exists u \text{ s.t. } |u| \leq p(|x|) \text{ and } M(x,y) = 1\}$$

is in **NP**.

Exercise 2. Prove that the following languages are in **NP**:

(a) Three colouring $\texttt{3COL} = \{\llcorner G \lrcorner : \text{ the graph } G \text{ has a colouring with three colours}\}$, where a colouring of G with c colours is an assignment of a number in $\{1, \ldots, c\}$ to each vertex such that no adjacent vertices get the same number.
(b) Graph isomorphism $\texttt{GRAPH\_ISOM} = \{(\llcorner G \lrcorner, \llcorner H \lrcorner) : \text{ the graphs } G \text{ and } H \text{ are isomorphic}\}$.

Exercise 3. Prove the remaining parts of Theorem 2.8:

(a) If the language L is **NP**-hard and $L \in \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.
(b) If the language L is **NP**-complete, then $L \in \mathbf{P}$ if and only if $\mathbf{P} = \mathbf{NP}$.

Exercise 4. Suppose $L_1, L_2 \in \mathbf{NP}$. Then is $L_1 \cup L_2$ in **NP**? What about $L_1 \cap L_2$?

Exercise 5. Let HALT be the Halting language (i.e., the language, whose characteristic function the halting function is). Show that HALT is **NP**-hard. Is it **NP**-complete?