Topology II

Exercise 5 (12.10.2017)

1. Denote $X \prec Y$, if the space X can be embedded in the space Y. Give an example, where $X \prec Y \prec X$, but $X \not\approx Y$. [Hint: One example can be obtained by considering intervals.]

2. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be the maps $f(x) = x$ and $g(x) = -x$. Prove, that the topology induced from the topology $\mathcal{T}_{pa}$ by the maps f and g is the discrete topology in $\mathbb{R}$.

3. Suppose, that in the set X we have the topology induced by the family $(f_j)_{j\in J}$, where $f_j\colon X \to Y_j$ and the spaces Y_j are Hausdorff spaces. Prove, that X is Hausdorff, if and only if for each pair $x, y \in X$, where $x \neq y$, there exists $j \in J$, such that $f_j(x) \neq f_j(y)$.

4. Suppose, that $f\colon X \to Y$ is continuous. Prove, that the *graph* of f,

$$\Gamma(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y,$$

is homeomorphic with X.

5. Let X be the product space $\mathbb{R}^{\mathbb{R}}$. Let $A \subset X$ be the set of all characteristic functions of finite sets.

a) Prove, that the constant map g, $g(x) = 1$, belongs to the closure of A.

b) Prove, that no sequence in A converges to g.

c) Deduce, that X is not metrizable.

6. Continuation to the previous exercise.
Construct a map $F\colon A \cup \{g\} \to \mathbb{R}$, which is not continuous, but for which $F(x_n) \to F(x)$, whenever $x_n \to x$ in the set $A \cup \{g\}$.