Topology II

Exercise 6 (17.10.2017)

1. Suppose, that $X = \prod_{j \in J} X_j$ is a product space, and $A_j \subset X_j$ for each $j \in J$. Then $A = \prod_{j \in J} A_j \subset X$.

a) Prove, that $\overline{A} = \prod_{j \in J} \overline{A_j}$.

b) Suppose, that A_j is a dense subset of X_j for each $j \in J$. Prove, that A is dense in X.

2. See Example 7.18.

a) Let $x = (x_1, x_2, ...) \in \{0, 1\}^{\mathbb{N}}$. Prove, that the sets

$$B_n(x) = \{(y_1, y_2, ...) \mid y_i = x_i \text{ for all } i = 1, ..., n\},$$

where $n \in \mathbb{N}$, form a neighbourhood basis at the point x.

b) Prove, that the map $f \colon X \to \mathbb{R}$ is continuous, directly from the definition of continuity. [Hint: You can use the result in a) above.]

c) Read through the proof, that f is injective.

3. Prove, that the map $\psi \colon I \to I^2$ in Example 7.18 is continuous.

4. a) Construct a continuous surjection $I \to I^3$.

b) Construct a continuous surjection $\mathbb{R} \to \mathbb{R}^2$.

5. Let $(X, \mathcal{T})$ be a space and $\mathcal{A}$ a collection of subsets of X, with relative topologies. The inclusions $j_A \colon A \to X$, where $A \in \mathcal{A}$, coinduce a topology $\mathcal{T}'$ to X.

a) Prove, that $\mathcal{T} \subset \mathcal{T}'$.

b) Prove, that $\mathcal{T}|A = \mathcal{T}'|A$ for all $A \in \mathcal{A}$.

c) Give an example, where $\mathcal{T} \neq \mathcal{T}'$.

6. Continuation to the previous exercise.
Prove, that $\mathcal{T} = \mathcal{T}'$, if $X = \mathbb{R}^n$ (with the usual topology) and $\mathcal{A}$ is the collection of all compact subsets.