REAL-VARIABLE HARMONIC ANALYSIS I
2017

3. Homework sheet
17.10.2017

3.1. **Homework.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. If $0 < \alpha < n$ and $\delta > 0$, show that there is a constant $c(n, \alpha)$ so that

$$\int_{B_\delta(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le c(n,\alpha)\delta^\alpha Mf(x) \text{ for all } x \in \mathbb{R}^n \,.$$

3.2. **Homework.** Let $1 < p < \infty$. Suppose that $f \in L^p(\mathbb{R}^n)$.
If $0 < \alpha p < n$ and $\delta > 0$ show that there is a constant $c(n, p, \alpha)$ so that

$$\int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \le c(n,p,\alpha)\delta^{\alpha-(n/p)} ||f||_{L^p} \,.$$

3.3. **Homework.** Let $1 < p < \infty$. Suppose that $f \in L^p(\mathbb{R}^n)$.
If $0 < \alpha p < n$ show, by using Homeworks 3.1 and 3.2 that there is a constant $c(n, p, \alpha)$ so that

$$\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \Big)^{p^*} dx \le c(n,p,\alpha) ||f||^{p^*}_{L^p(\mathbb{R}^n)} \,.$$

Here, $p^* = np/(n - \alpha p)$.
Lars-Inge Hedberg (around 1972) proved this inequality by using the Hardy-Littlewood maximal inequality. The method is called Hedberg's method.

3.4. **Homework.** Let $1 < p < n$. Suppose that $f \in C_0^\infty(\mathbb{R}^n)$. By using Homework 3.3 prove the Sobolev inequality: there exists a constant $c = c(n, p)$ so that

$$||f||_{L^{p^*}((\mathbb{R}^n)} \le c||\nabla f||_{L^p(\mathbb{R}^n)} \,.$$

First show, by starting the indentity

$$f(x) = -\int_0^\infty \partial_r f\big(x + r\frac{x-y}{|x-y|}\big)\, dr$$

that

$$|f(x)| \le c(n) \int_{\mathbb{R}^n} \frac{|\nabla f(y)|}{|x-y|^{n-1}}\, dy.$$