Tree property for inaccessible κ implies weak compactness

We follow Jech. Suppose κ is (strongly) inaccessible and has the tree property. Suppose $F : [\kappa]^2 \to 2$ is a partition.

We build a tree T of height κ with levels of size $< \kappa$. The tree consists of binary sequences t of some length γ, where always $\gamma < \kappa$. The sequences of length γ form the γ'th level of the tree T. So each level is of card $< \kappa$ because branching is binary and at limits we use strong inaccessibility.

The sequences are constructed by induction. Suppose some sequences t_ξ, $\xi < \alpha$, have been already constructed. We assume $t_0 = \emptyset$. We now construct $t_\alpha(\xi)$ by induction on $\xi < \kappa$. The construction will end at some stage $\xi < \kappa$. At first we let $t_\alpha(0) = F(\{0, \alpha\})$. Then $t_\alpha(1) = F(\{\beta, \alpha\})$, where $t_\alpha \restriction 1 = t_\beta$, if such a β can be found, otherwise the construction of t_α ends. In general, if $t_\alpha(\xi)$, $\xi < \zeta$, has been defined, we look for β such that $t_\alpha \restriction \zeta = t_\beta$ and if found let $t_\alpha(\zeta) = F(\{\beta, \alpha\})$. If not found, we consider t_α constructed.

The tree has size κ: Whenever we construct t_α, $\alpha < \kappa$, it is always a new binary sequence. So we get at least κ elements for T.

Levels are of size $< \kappa$. For a length $\gamma < \kappa$ there are only $2^{|\gamma|}$ binary sequences of length γ. And we assumed κ is strongly inaccessible.

By the tree property, T has a branch B of length κ. Let

$$H_0 = \{\alpha < \kappa : \text{ both } t_\alpha \in B \text{ and } t_\alpha \frown 0 \in B\}$$

$$H_1 = \{\alpha < \kappa : \text{ both } t_\alpha \in B \text{ and } t_\alpha \frown 1 \in B\}.$$

By the Pigeonhole Principle, either H_0 or H_1 has cardinality κ, say it is H_0. We show that H_0 is homogeneous, in fact that $F(\{\alpha, \beta\}) = 0$ for all $\alpha \neq \beta$ in H_0. Suppose t_α has length γ and t_β has length δ. Since both are on the branch B, they cannot have the same length. So one is bigger, say $\delta > \gamma$. Recall how $t_\beta(\gamma)$ was defined. Since they are on the same branch, $t_\beta \restriction \gamma = t_\alpha$. By the construction, $t_\beta(\gamma) = F(\{\beta, \alpha\})$. Since $t_\alpha \frown 0 \in B$ and B is a branch, $t_\beta(\gamma) = 0$. Hence $F(\{\beta, \alpha\}) = 0$. QED