This is to accompany section 4.2 of the book.

Linear stability analysis

Consider a system of ODEs

$$\frac{dX}{dt} = f(X)$$

with $X(t) \in \mathbb{R}^n$ and $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. Let $\hat{X}$ be an equilibrium of this system ($f(\hat{X}) = 0$) and let $x(t) = X(t) - \hat{X}$ denote the difference between the current state and the equilibrium. For the difference, we have $\frac{dx}{dt} = \frac{dX}{dt} = f(x + \hat{X})$ and as long as the difference is small, we can Taylor expand $f(x + \hat{X}) = f(\hat{X}) + Jx + O(||x||^2)$ to arrive at the linearized dynamics

$$\frac{dx}{dt} = Jx \tag{1}$$

where J is the Jacobian matrix with elements $J_{ij} = [\partial f_i(X)/\partial X_j]_{X=\hat{X}}$. To solve the linearized dynamics in (1), assume[1] that J has n linearly independent eigenvectors $u_1, ..., u_n$, and write $x(t)$ in the base of the eigenvectors,

$$x(t) = \sum_{i=1}^{n} y_i(t) u_i$$

or equivalently

$$x(t) = Uy(t) \tag{2}$$

where U is a matrix the columns of which are the eigenvectors u_i, and $y(t)$ is the vector with elements $y_i(t)$. Substituting (2) into 1 yields

$$\frac{dy}{dt} = U^{-1} J U y$$

Let Λ be the diagonal matrix with elements λ_i, the eigenvalues of J. By the definition of the eigenvalues and eigenvectors, we have $JU = U\Lambda$, so that $U^{-1}JU = \Lambda$ and we arrive at the decoupled system

$$\frac{dy}{dt} = \Lambda y \quad \Longleftrightarrow \quad \frac{dy_i}{dt} = \lambda_i y_i \quad \text{for } i = 1, ..., n \tag{3}$$

[1] This is a mild assumption. The results are valid also if this assumption does not hold, but the proof is longer.

This we solve easily,

$$y_i(t) = y_i(0)e^{\lambda_i t}$$

so that the difference from the equilibrium grows or declines according to

$$x(t) = \sum_{i=1}^{n} y_i(0)e^{\lambda_i t}u_i \tag{4}$$

The equilibrium is asymptotically stable if $\lim_{t\to\infty} x(t) = 0$ for every initial vector $x(0)$, or, equivalently, for every $y(0)$. From (4), it is obvious that every real eigenvalue should be negative for stability; if $\lambda_i > 0$, then the term $y_i(0)e^{\lambda_i t}u_i$ explodes to infinity for all initial vectors with $y_i(0) \neq 0$ (i.e., the system explodes in the direction of the eigenvector u_i).

To investigate the case of complex eigenvalues, first notice that if

$$\lambda_1 = \alpha + i\beta$$

satisfies the characteristic equation of J (which has only real coefficients), then also

$$\lambda_2 = \alpha - i\beta$$

satisfies the characteristic equation (the imaginary part will cancel with either $+$ or $-$ in front of it). Hence complex eigenvalues come in complex conjugate pairs. The eigenvectors u_1 and u_2 are also complex conjugates, and in (2), $y_1(t)$ and $y_2(t)$ are complex conjugates. It follows that the two vectors $y_1(0)u_1$ and $y_2(0)u_2$ are also complex conjugates and we can write them as

$$\begin{aligned} y_1(0)u_1 &= \gamma + i\delta \\ y_2(0)u_2 &= \gamma - i\delta \end{aligned}$$

with some suitable (real) vectors γ and δ. Substituting into (4) and using Euler's formula for the complex exponentials, we obtain

$$\begin{aligned} x(t) &= e^{\lambda_1 t}y_1(0)u_1 + e^{\lambda_2 t}y_2(0)u_2 + ... \\ &= e^{(\alpha+i\beta)t}(\gamma + i\delta) + e^{(\alpha-i\beta)t}(\gamma - i\delta) + ... \\ &= e^{\alpha t}\left[\left(e^{i\beta t} + e^{-i\beta t}\right)\gamma + \left(e^{i\beta t} - e^{-i\beta t}\right)i\delta\right] + ... \\ &= 2e^{\alpha t}\left[\cos(\beta t)\gamma - \sin(\beta t)\delta\right] + ... \end{aligned} \tag{5}$$

where '...' stands for the terms $i = 3, ..., n$ in (4). First of all, notice that the result for $x(t)$ is a real vector (gooood). The terms corresponding to the complex conjugate pair of eigenvalues we currently investigate will decay to zero if α, the real part of $\lambda_{1,2}$, is negative; hence the condition for asymptotic stability is that *each eigenvalue of the Jacobian must have negative real part.*

The relaxation time is defined for the scalar decay process $x(t) = x(0)e^{\alpha t}$ (with $\alpha < 0$) as the time needed for x to decay by a factor e^{-1} (i.e., t such that $x(t) = e^{-1}x(0)$), which is $t = 1/|\alpha|$. By analogy, also in a planar system with two complex conjugate eigenvalues the relaxation time is taken to be $1/|\alpha| = 1/|Re(\lambda)|$.

The expression in the brackets of (5), $[\cos(\beta t)\gamma - \sin(\beta t)\delta]$, is a periodic function of t with period $T = 2\pi/\beta$. After time T, the vectors γ and δ get the same coefficients, barring the common, decaying or exploding factor $e^{\alpha t}$ in (5). Hence the solution $X(t) = \hat{X} + x(t)$ will deviate from the equilibrium $\hat{X}$ in the same direction at t and at $t + T$. The complex conjugate pair of eigenvalues corresponds to a decaying or exploding oscillation ("circling") around the equilibrium, where $\alpha = Re(\lambda)$ gives the speed of decay/growth, and $\beta = Im(\lambda)$ controls the period of the oscillations, $T = 2\pi/\beta$.