Dependence Logic
Exercise 7

1. Let $k \geq 1$ and R a k-ary relation symbol. Construct a ESO(k-ary) sentence ϕ of vocabulary $\tau = \{\leq, R\}$ such for all finite ordered τ-structures $\mathcal{M}$:

$$\mathcal{M} \models \phi \Leftrightarrow |R^{\mathcal{M}}| \text{ is even.}$$

($\mathcal{M}$ being a finite ordered structure means that $|\mathrm{Dom}(\mathcal{M})| = n$ for some $n \in \mathbb{N}$ and $\leq^{\mathcal{M}}$ is a linear order of $\mathrm{Dom}(\mathcal{M})$.)

2. A sentence $\psi \in \Sigma^1_1[\tau \cup \{R\}]$ is said to be downwards monotone with respect to the relation R if for all τ-structures $\mathcal{M}$ and interpretations P, P' for R

$$\text{if } (\mathcal{M}, P) \models \psi, \text{ and } P' \subseteq P, \text{ then } (\mathcal{M}, P') \models \psi.$$

Show using induction on $\phi \in CL_{FO^*}(\Sigma^1_1)$ that if R appears only negatively in ϕ (i.e., all occurrences are of the form $\neg R(\vec{t})$), then ϕ is downwards monotone with respect to R.

3. Let a sentence $\psi \in \Sigma^1_1[\tau \cup \{R\}]$ be downwards monotone with respect to R. Show that then ψ is logically equivalent with the sentence

$$\exists R' \big(\forall \vec{x}(\neg R(\vec{x}) \vee R'(\vec{x})) \wedge \psi(R'/R) \big)$$

in which R appears only negatively.

4. Let $\phi := \forall y \forall z \theta$ be an independence logic formula where θ is a quantifier free formula. Show that there exists a formula $\phi' := \forall y \exists z \theta' \in \mathrm{FO}(\perp_c, \subseteq)$ where θ is quantifier free such that $\phi \equiv \phi'$.