Infinite games

We work first in the real interval $[0,1]$. Let $A \subseteq [0,1]$. In the game $G(A)$ the players contribute an infinite sequence

$$(n_0, n_1, n_2, ...),$$

where $n_i \in \{0, ..., 9\}$, and player I wins if the real

$$0.n_0 n_1 ...$$

is in A. Player I plays $n_0, n_2, ...$ and II plays $n_1, n_3, ...$

Claim: Player I has a winning strategy in $G([0, 0.5])$.

Proof: Player I plays e.g. $n_0 = 2$ and the resulting real is in $[0, 0.5]$, whatever II tried to do.

Claim: Player II has a winning strategy in $G(\{0.3\})$.

Proof: Player I can try e.g. $n_0 = 3$ (or anything else) but II can play n_1 so that the resulting real is not 0.3, whatever II tried to do.

Claim: Player II has a winning strategy in $G([0,1] \cap Q)$, where Q is the set of rationals.

Proof: Player II can enumerate all rationals and play in such a way that they are all avoided.

We work now in the Baire space ω^ω. Let $A \subseteq \omega^\omega$. In the game $G(A)$ the players contribute an infinite sequence

$$(n_0, n_1, n_2, ...),$$

where $n_i \in \omega$ and player I wins if the sequence is in A. Player I plays $n_0, n_2, ...$ and II plays $n_1, n_3, ...$

Claim: If A is closed then $G(A)$ is determined.

Proof: If player II has a winning strategy, we are done. Suppose II does not have a winning strategy. We describe a winning strategy of I. Player I must have a first move n_0 such that II does not have a winning strategy for the rest of the game, because otherwise, whatever I moves, II has a winning strategy, meaning that II has a winning strategy already before I moves n_0. After I moves this n_0, player II moves something n_1. There must be **some** continuation of (n_0, n_1) to an infinite sequence $(n_0, n_1, n_2, ...)$ in A, for otherwise II "can take a nap".

Again, player I must have a move n_2 such that II does not have a winning strategy for the rest of the game, because otherwise, whatever I moves, II has a winning strategy, meaning that II has a winning strategy already before I moves n_2. So II moves this n_2 and then I moves something n_3. Again, there must be **some** continuation of (n_0, n_1, n_2, n_3) to an infinite sequence $(n_0, n_1, n_2, ...)$ in A, for otherwise II "can take a nap".

We continue like this and produce $s = (n_0, n_1, n_2, ...)$. Now we argue that $s \in A$ i.e. that I wins. Suppose $s \notin A$. Since A is closed, there is an open neighborhood N of s such that $N \cap A = \emptyset$. Thus there is an i such that every continuation of $(n_0, n_1, ..., n_i)$ is outside of A. W.l.o.g. this i is odd. By construction, there is **some** continuation of $(n_0, n_1, ..., n_i)$ to an infinite sequence $(n_0, n_1, n_2, ...)$ in A, a contradiction. QED

By symmetry:

Claim: If A is open then $G(A)$ is determined. QED

Claim: There is A such that $G(A)$ is non-determined.

List as $\{f_\alpha : \alpha < 2^\omega\}$ all possible strategies of I, and as $\{g_\alpha : \alpha < 2^\omega\}$ all possible strategies of II. Thus f_α maps sequences $(n_0, n_1, ..., n_{2i-1})$ to a move n_{2i} of I, and g_α maps sequences $(n_0, n_1, ..., n_{2i})$ to a move n_{2i+1} of II. We construct a set A of plays $s = (n_0, n_1, n_2, ...)$ such that for each α, neither f_α not g_α is a winning strategy.

We define disjoint sets $\{a_\alpha : \alpha < 2^\omega\}$ and $\{b_\alpha : \alpha < 2^\omega\}$. We choose $b_\alpha \notin \{a_\beta : \beta < \alpha\}$ so that b_α follows the strategy f_α. This is possible since we can choose the moves of II arbitrarily so there are 2^ω such plays. We choose $a_\alpha \notin \{b_\beta : \beta \leq \alpha\}$ so that a_α follows the strategy g_α. This is possible since we can choose the moves of I arbitrarily so there are 2^ω such plays. Let $A = \{a_\alpha : \alpha < 2^\omega\}$.

$G(A)$ is non-determined: f_α is not a winning strategy of I since b_α follows f_α and still $b_\alpha \notin A$. g_α is not a winning strategy of II since a_α follows g_α and still $a_\alpha \in A$.

Finally, let us work on ω_1. This time we decide that if $A \subseteq \omega_1$ and the players contribute $(\alpha_0, \alpha_1, ...)$, where $\alpha_i \in \omega_1$, then player I wins if $\sup_n \alpha_n$ is in A and otherwise player II wins.

Claim: If A contains a club C, then I has a winning strategy in $G(A)$, and conversely.

Proof: The strategy of I is to always play a bigger element than the elements played so far, and always in C. Since C is closed, he wins. On the other hand, if I has a winning strategy f, then A contains the club of ordinals β such that if $\alpha_0, ..., \alpha_n < \beta$, then $f(\alpha_0, ..., \alpha_n) < \beta$. QED

Claim: If A is disjoint from a club C, then II has a winning strategy in $G(A)$.

Proof: The strategy of II is to always play a bigger element than the elements played so far, and always in C. Since C is closed, she wins. On the other hand, if II has a winning strategy g, then A contains the club of ordinals β such that if $\alpha_0, ..., \alpha_n < \beta$, then $g(\alpha_0, ..., \alpha_n) < \beta$. QED

Conclusion: A is stationary iff II does not have a winning strategy. $-A$ is stationary iff I does not have a winning strategy. A is bi-stationary iff $G(A)$ is non-determined.