Topology II

Exercise 8 (9.11.2017)

1. Prove, that the projective space presented in Example 9.12.6 is homeomorphic with the space P^2 presented in item 9.5. Hint: At first replace I^2 with $\bar{B}^2$ as in 9.12.6. Then define an embedding $f\colon \bar{B}^2 \to S^2$ to the upper hemisphere, such that $f(x, y) = (x, y, 0)$ if $(x, y) \in S^1$. Then apply exercise 5 of last week.

2. Consider the quotient group $\mathbb{R}/\mathbb{Q} = \{x + \mathbb{Q} \mid x \in \mathbb{R}\}$. Prove, that the quotient topology is the mini-topology.

3. Let U be an open set in a metric space (X, d). Construct a continuous function $f\colon X \to \mathbb{R}$, such that $U = \{x \in X \mid f(x) > 0\}$. Hint: Remember that $d(x, A)$ is a continuous function of x.

4. A subset A of a space X is called a G_δ set, if it can be presented as a countable intersection of open sets. A subset A is called an F_σ set, if it can be presented as a countable union of closed sets. Prove, that in a metric space, every closed set is a G_δ set, and every open set is an F_σ set. Hint: Consider sets like $\{x \mid d(x, A) < 1/n\}$.

5. Let $f\colon \mathbb{R}^n \to \mathbb{R}^p$ be a continuous surjection. Prove, that there exists $r > 0$, such that the set $fB(\bar{0}, r)$ has interior points. Hint: Baire's theorem and compactness.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, which has derivatives of all orders. Suppose, that for every $x \in \mathbb{R}$ there exists $n(x) \in \mathbb{N}$, such that the derivative $f^{(n(x))}(x) = 0$. Prove, that every interval $[a, b]$ contains an interval $]c, d[$, where f is a polynomial. Hint: Use Baire's theorem and the sets $A_k = \{x \in \mathbb{R} \mid f^{(k)}(x) = 0\}$.