Dependence Logic
Exercise 8

1. Let a quantifier $\forall^1$ be defined as follows:

$$\mathcal{M} \models_X \forall x^1 \psi \text{ if and only if, for all } a \in \mathrm{Dom}(\mathcal{M}) : \ \mathcal{M} \models_{X[F_a/x]} \psi,$$

where $F_a(s) = \{a\}$ for all $s \in X$. Denote by FO$(=(\ldots), \forall^1)$ the extension of dependence logic with the quantifier $\forall^1$. Show that the formulas of FO$(=(\ldots), \forall^1)$ have the downward closure property (see Proposition 21 of the lecture notes).

2. Extend the translation (Theorem 45 of the lecture notes) of independence logic formulas to Σ^1_1 to cover also the case of $\forall^1$.

3. Let M be a set and $\Phi \colon \mathcal{P}(M) \to \mathcal{P}(M)$.

- The function Φ is *monotone* if for all $P \subseteq P' \subseteq M$, $\Phi(P) \subseteq \Phi(P')$.
- The function Φ is *inflationary* if for all $P \subseteq M$, $P \subseteq \Phi(P)$.
- The function Φ is *inductive* if its stages satisfy $\Phi_\alpha \subseteq \Phi_\beta$ for all ordinals $\alpha \leq \beta$, where the stages of Φ are defined by: $\Phi_0 := \emptyset$, $\Phi_{\alpha+1} := \Phi(\Phi_\alpha)$, and $\Phi_\lambda := \cup_{\alpha<\lambda} \Phi_\alpha$ for a limit ordinal λ.

Show that if Ψ is either monotone or inflationary then it is inductive. You may assume M to be finite if you have not studied transfinite induction before.

4. Give examples of inductive operators Φ_1 and Φ_2 such that Φ_1 is monotone but not inflationary and Φ_2 is inflationary but not monotone.