This is a more detailed solution to exercise 2.6 of the book.

1 The fundamental matrix solution of homogeneous linear ODE systems

Consider the homogeneous linear system of n ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} \tag{1}$$

with the initial condition $\mathbf{x}(0) = \mathbf{x_0}$. Below we shall take $x_i(t)$ to be the number of individuals in state i at time t, and then $\mathbf{A}$ is the matrix of transition rates. What is in this section, however, applies to any homogeneous linear ODE system.

Define $e^{t\mathbf{A}}$ as a matrix given by

$$e^{t\mathbf{A}} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$$

(this is the matrix version of the power series that defines the exponential function). A simple substitution verifies that the solution of (1) is

$$\mathbf{x}(t) = e^{t\mathbf{A}}\mathbf{x_0} \tag{2}$$

but of course this is no real help because computing $e^{t\mathbf{A}}$ from its definition is not practical.

A fundamental matrix solution is a matrix $\mathbf{M}(t)$ that contains n linearly independent solutions of (1) as its columns. This means that every solution can be obtained as a linear combination of the columns of $\mathbf{M}(t)$, i.e.,

$$\mathbf{x}(t) = \mathbf{M}(t)\mathbf{c} \tag{3}$$

where $\mathbf{c}$ is a constant vector that depends on the initial condition. By (2), $e^{t\mathbf{A}}$ is a fundamental matrix solution with $\mathbf{c} = \mathbf{x_0}$.

If $\mathbf{A}$ is diagonalizable, i.e., if it has n linearly independent eigenvectors $\mathbf{u_1}, ..., \mathbf{u_n}$, then a useful fundamental matrix solution is

$$\mathbf{M}(t) = \left[e^{\lambda_1 t}\mathbf{u_1}, ..., e^{\lambda_n t}\mathbf{u_n}\right] \tag{4}$$

where $\lambda_1, ..., \lambda_n$ are the eigenvalues. Again, a simple substitution verifies that each column of $\mathbf{M}(t)$ is a solution of (1), and the columns are independent because the eigenvectors are independent. To find $\mathbf{c}$ in (3), note that $\mathbf{x}(0) = \mathbf{M}(0)\mathbf{c} = \mathbf{x_0}$, and hence $\mathbf{c} = \mathbf{M}(0)^{-1}\mathbf{x_0}$. Notice that this paragraph is nothing but a concise presentation of the diagonalization procedure that we used to prove the conditions for linear stability of an equilibrium.

2 The expected time spent in various states of a continuous-time Markov chain

Let us know take $x_i(t)$ to be the number of individuals in state i at time t, and $\mathbf{A}$ to be the matrix of transition rates that specify how the individuals move between the states or move out of all states by death (or recovery when considering disease dynamics). Importantly, $\mathbf{A}$ does not include reproduction (or new infections), only transitions that already existing individuals (or individuals already infected) are subject to. Because death is always a possible transition, in biologically relevant models we have that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$, because after infinitely long time, all individuals are dead. Note that by (2), the assumption $\lim_{t->\infty} e^{t\mathbf{A}} = \mathbf{0}$ is equivalent to $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ for all initial vectors $\mathbf{x_0}$.

To calculate the expected time spent in state i, start with the fact that there are $x_i(t)$ individuals who spend dt in state i between time t and $t + dt$. Integrating for all times gives the total time spent by all individuals in state i as $\int_0^\infty x_i(t)dt$. Finally, dividing with the total number of individuals $||\mathbf{x_0}||$ (where the norm is the sum norm) gives the average time spent by one individual in state i before he dies,

$$\frac{\int_0^\infty x_i(t)dt}{||\mathbf{x_0}||}$$

Writing the above for the entire vector $\mathbf{x}(t)$ and substituting $\mathbf{x}(t)$ from (2) yields

$$\frac{\int_0^\infty \mathbf{x}(t)dt}{||\mathbf{x_0}||} = \int_0^\infty e^{t\mathbf{A}} dt \; \frac{\mathbf{x_0}}{||\mathbf{x_0}||}$$

If we consider one individual initially in state j, then $\mathbf{x_0} = \mathbf{e_j}$ and the expected time spent in state i is given by the (i, j) element of the matrix $\int_0^\infty e^{t\mathbf{A}} dt$.

To make sense of this result, we need the following

Lemma. If $\lim_{t->\infty} e^{t\mathbf{A}} = \mathbf{0}$, then $\int_0^\infty e^{t\mathbf{A}} dt = -\mathbf{A}^{-1}$.

Proof. The claim is equivalent to $\int_0^\infty e^{t\mathbf{A}} \mathbf{A} dt = -\mathbf{I}$. Substitute the definition of $e^{t\mathbf{A}}$, integrate, and use $\lim_{t->\infty} e^{t\mathbf{A}} = \mathbf{0}$ in the last step:

$$\begin{aligned}
\int_0^\infty e^{t\mathbf{A}} \mathbf{A} dt &= \int_0^\infty \sum_{k=0}^\infty \frac{t^k}{k!} \mathbf{A}^{k+1} dt = \\
&= \left[\sum_{k=0}^\infty \frac{t^{k+1}}{(k+1)!} \mathbf{A}^{k+1} \right]_0^\infty = \left[\sum_{k=1}^\infty \frac{t^k}{k!} \mathbf{A}^k \right]_0^\infty = \left[e^{t\mathbf{A}} - \mathbf{I} \right]_0^\infty = \\
&= -\mathbf{I}
\end{aligned}$$

Therefore we conclude that starting from state j, the expected time an individual will spend in state i is given by $(-\mathbf{A}^{-1})_{ij}$.

Notice that if there is only one state, then $\mathbf{A}$ is a 1×1 matrix the only element of which is the negative of the death rate, $-\mu$. Hence we recover the well known result that the expected lifetime of an individual with a constant death rate μ is given by $1/\mu$.