Problem set 2: Ultraproducts!

Problem 1: Prove that if $\mathcal{U}$ is a principal ultrafilter on ω, then $\mathbb{N}^\omega/\mathcal{U}$ is isomorphic to $\mathbb{N}$.

Problem 2: Show that if M is an infinite model and $\mathcal{U}$ is a nonprincipal ultrafilter on a set S, then M is isomorphic to a proper elementary submodel of $\prod_{x\in S} M_x/\mathcal{U}$. (In the notation of the problem, $M_x = M$.)

Problem 3: Prove that if M_n is a graph which is a cycle of $n+3$ elements, and $\mathcal{U}$ is a nonprincipal ultrafilter on ω, then the ultraproduct $\prod_{n\in\omega} M_n/\mathcal{U}$ is a disconnected graph.

Problem 4: Prove that if M is the transitive collapse of the ultraproduct of V with respect to a κ-complete nonprincipal ultrafilter on κ, then M is closed under κ sequences.

Problem 5: Prove that if i is the embedding of V into the inner model M of the above problem 4, then $2^\kappa < i(\kappa) < (2^\kappa)^+$. (QUESTION: How is it possible that $i(\kappa)$ is between a cardinal and its successor, even though $i(\kappa)$ is a cardinal in M?)