Dependence Logic
Exercise 9

1. Let M be a set and $\Phi\colon \mathcal{P}(M^k) \to \mathcal{P}(M^k)$ a monotone operator. Show that

$$\mathrm{gfp}(\Phi) = \{X \subseteq M^k \mid X \subseteq \Phi(X)\}.$$

2. Let $\mathcal{M}$ be a finite graph over vocabulary $\{E\}$. Define an operator $\Phi\colon \mathcal{P}(\mathrm{Dom}(\mathcal{M})^2) \to \mathcal{P}(\mathrm{Dom}(\mathcal{M})^2)$ as follows:

$$\Phi(R) := \{(s(x), s(y)) \mid (\mathcal{M}, R) \models_s \psi\},$$

where

$$\psi := R(x, y) \vee E(x, y) \vee \exists z(E(x, z) \wedge R(z, y)).$$

Show using induction on n that the finite stages Φ^n of Φ satisfy the following for $n \geq 1$:

$$\Phi^n = \{(a, b) \in \mathrm{Dom}(\mathcal{M})^2 | \text{ the distance of } a \text{ and } b \text{ in } \mathcal{M} \text{ is at most } n\}.$$

Does Φ have fixed points other than the transitive closure of $E^{\mathcal{M}}$?

3. Let $\psi \in \mathrm{FO}[\tau \cup \{R\}]$ and let $\vec{x}$ be a tuple of variables such that $ar(R) = |\vec{x}| = k$. Assume also that R appears only positively in ψ. Let $\mathcal{M}$ be a τ-structure. Define an operator $F_\psi\colon \mathcal{P}(\mathrm{Dom}(\mathcal{M})^k) \to \mathcal{P}(\mathrm{Dom}(\mathcal{M})^k)$ as follows:

$$F_\psi(R) := \{s(\vec{x}) \mid (\mathcal{M}, R) \models_s \psi\}.$$

Show that F_ψ is a monotone operator. You may assume that ψ is (upwards) monotone with respect to R since R appears only positively in it.

4. Let Φ be a k-ary monotone operator on M and $\tilde{\Phi}$ its dual. Show that $\tilde{\Phi}$ is also monotone and that $\mathrm{gfp}(\Phi) = M^k \setminus \mathrm{lfp}(\tilde{\Phi})$ (see Proposition 71 of the lecture notes).