Topology II

Exercise 9 (16.11.2017)

1. Prove the Weierstrass criterion (Proposition 10.14). Hint: Denote $R_n = \sum_{j=n+1}^{\infty} M_j$ and $s_n = \sum_{j=1}^{n} u_j$. Prove first, that for all $x \in D$ and $p \in \mathbb{N}$ we have, that $|s_{n+p}(x) - s_n(x)| \leq R_n$, and then use completeness.

2. Let (X, d) be a metric space. The *Hausdorff distance* of two non-empty bounded subsets A and B is

$$D(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}.$$

Prove, that D is a pseudo-metric in the set $Y = \{A \mid \emptyset \neq A \subset X,\ A \text{ is bounded}\}$, and D is a metric in the set $Y_0 = \{F \in Y \mid F \text{ is closed}\}$.

3. a) Prove, that the properties T_0, T_1, T_2, T_3 are hereditary (Proposition 11.9).

b) Try to prove that the property T_4 is hereditary analogously as you proved that T_3 is hereditary. In which step do you get a problem?

4. Prove, that a space X is Hausdorff, if and only if the diagonal $\Delta = \{(x, x) \mid x \in X\}$ is a closed subset of the product space $X \times X$.

5. Suppose, that $f\colon X \to Y$ is continuous, where Y is a Hausdorff space. Prove, that the graph $\Gamma = \{(x, f(x)) \mid x \in X\}$ is closed in $X \times Y$. Using the previous exercise, deduce, that this result doesn't hold in general without the assumption, that Y is Hausdorff.

6. Prove, that the space $(\mathbb{R}, \mathcal{T}_{pa})$ is normal. Hint: Let A and B be disjoint closed sets. For each $x \in A$ choose an interval $[x, x + r(x)[$, which doesn't meet B. Then the union of these sets is a neighbourhood of A. Construct similarly a neighbourhood for B, and prove, that these neighbourhoods are disjoint.