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WEAKLY COMPACT CARDINALS: A COMBINATORIAL PROOF

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We give here a direct purely combinatorial proof that weak compactness is equivalent to a combinatorial property (2). This property (2) is apparently stronger, and from it, all other usual equivalent definitions and usual properties of weakly compact cardinals can be deduced. So this proof may be useful for books which want to present weakly compact cardinals, but not logic.

A direct simple proof of a weaker implication (e.g., weakly compact μ is not the first inaccessible, and every stationary set has an initial segment which is stationary) was given by Kunen [K], and independently by the author [Sh]. Baumgartner [B] had another proof.

We were motivated by the manuscript of Erdös, Hajnal, Mate and Rado's book on partition calculus, and by conversations with A. Levi who was writing a book on naïve set theory.

Notation. Let $i, j, \alpha, \beta, \gamma, \eta, \sigma$ be ordinals, μ be a cardinal, f, g be functions.

Let $\mathrm{cf}\,\alpha$ be the cofinality of α.

A partially ordered set T is a tree if for any $a \in T$, $\{b : b < a, b \in T\}$ is well ordered; its order type (an ordinal) is called the level of a, and T_α is the set of $a \in T$ of level α.

A tree T is a μ-tree if: T has an element of level α iff $\alpha < \mu$ and $|T_\alpha| < \mu$ for every $\alpha < \mu$.

A branch of a tree T is a maximal, totally ordered subset. A μ-branch is a branch of order type μ.

REMARK 1. For μ the first inaccessible we can define a μ-tree with no μ-branch by:

$$T = \{h\colon \mathrm{Dom}\, h \text{ an ordinal } \alpha < \mu,\ h(i) < 1 + i, \text{ and for strong limit } i, j \in \mathrm{Dom}\, h,\ h(i) \neq h(j)\}.$$

THEOREM 1. *For μ strongly inaccessible the following are equivalent*:

(1) *μ is weakly compact, i.e., every μ-tree has a μ-branch.*

(2) *For every family of functions $f_\alpha\colon \alpha \to \alpha$ $(\alpha < \mu)$ there is a function $f : \mu \to \mu$ such that:* $(\forall \alpha < \mu)(\exists \beta)[\alpha \le \beta < \mu$ *and* $f_\beta \restriction \alpha = f \restriction \alpha]$.

REMARK 2. It is easy to prove (2) implies Baumgartner principle implies (1).

REMARK 3. We now show it is easy to deduce from the theorem that weakly compact cardinals are large.

Let I be the family of subsets S of μ such that (2) is not satisfied if we replace "$(\forall \alpha < \mu)$ $(\exists \beta)$..." by "$(\forall \alpha < \mu)(\exists \beta \in S)$...". Clearly it suffices to define f_j for $j \in S$. So by the theorem, $\mu \in I$ iff I is weakly compact. It is trivial that $S' \subseteq S \in I \Rightarrow$

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$S' \in I$. Now, I is closed under union of $< \mu$ elements. Because if $S_\xi \in I$ $(\xi < \alpha < \mu)$, let f_j^ξ $(j < \lambda)$ exemplify $S_\xi \in I$ [i.e., there is no $f: \mu \to \mu$ such that $(\forall \alpha < \mu)(\exists \beta \in S_\xi) \cdot [(\alpha \leq \beta < \mu$ and $f_\beta^\xi \restriction \alpha = f \restriction \alpha)]$. Define: f_j is f_j^ξ if $j \in S_\xi - \bigcup_{\zeta<\xi} S_\zeta$, and f_j^0 otherwise. If $\bigcup_{\xi<\alpha} S_\xi \notin I$ then there are $f: \mu \to \mu$ and, for each $\alpha < \mu$, $\beta(\alpha) \in \bigcup_{\xi<\alpha} S_\xi$, $\alpha \leq \beta(\alpha) < \mu$, $f_{\beta(\alpha)} \restriction \alpha = f \restriction \alpha$. As μ is regular there is an unbounded $A \subset \mu$, and $\xi < \alpha$ such that $\alpha \in A \Rightarrow \beta(\alpha) \in S_\xi - \bigcup_{\gamma<\xi} S_\zeta$. Defining $\beta'(\alpha)$ as $\beta(\alpha')$ for the first α', $\alpha \leq \alpha' \in A$, we see that f, $\beta'(\alpha)$ contradict the choice of $f_j = f_j^\xi$ (for $j \in S_\xi - \bigcup_{\zeta<\xi} S_\zeta$. In fact I is normal, i.e., $S_\xi \in I$ $(\xi < \mu)$ implies $S = \{\alpha : \alpha \in \bigcup_{\zeta<\alpha} S_\zeta\} \in S$. [Let f_α^ξ $(\alpha \in S_\xi)$ exemplify $S_\xi \in I$, $\zeta(\alpha) < \alpha$ $(\alpha \in S)$ be such that $\alpha \in S_{\zeta(\alpha)}$. Define f_σ $(\sigma \in S)$ by: $f_\sigma(0) = \zeta(\sigma)$, $f_\sigma(1 + i) = f_\sigma^{\zeta(\sigma)}(i)$.]

Suppose μ is weakly compact. Then I is a μ-complete filter over I. If $S \subseteq \mu$ is closed unbounded, define f_σ $(\sigma \notin S)$ by $f_\sigma(\alpha) = \max(S \cap \sigma)$, hence for every α, $\beta < \mu$, $\{\sigma : f_\sigma(\alpha) = \beta, \sigma \in S\}$ is a bounded subset of μ, hence the f_σ's exemplify $\mu - S \in I$. Let S be the set of singular ordinals $< \mu$, and for each $\sigma \in S$ let $\sigma = \sum_{i<\mathrm{cf}\,\sigma} \sigma_i$, where cf σ, $\sigma_i < \sigma$, and define $f_\sigma(1 + \alpha) = \min\{\sigma_i : \sigma_i \leq \alpha\}$ (for $\alpha < \sigma \in S$) and $f_\sigma(0) = \mathrm{cf}\,\sigma$.

From the above it is clear that any closed unbounded subset of μ contains an inaccessible cardinal, i.e., μ is Mahlo. Also, the not-Mahlo cardinals etc. are in I.

Conclusion 4. (1) If μ is strongly inaccessible, $S \subseteq \mu$ is stationary but $\delta \cap S$ is not stationary for every $\delta < \mu$ *then* μ is not weakly compact.

(2) If μ is strongly inaccessible, $S_\alpha \subseteq \mu$ stationary (for $\alpha < \mu$) and for every strongly inaccessible $\sigma < \mu$ for some $\beta < \sigma$, $S_\beta \cap \sigma$ is not stationary then μ is not weakly compact.

PROOF OF 4. (1) Let $f_\sigma: \sigma \to \sigma$ enumerate a closedun bounded subset of σ disjoint to S, f_σ monotonic. If μ is weakly compact, by (3) we have f as in 1(2), which shows S is not stationary.

(2) Similar.

PROOF OF THEOREM 1. Trivially (2) implies (1). Let us prove the other direction. Let for $\alpha \leq \mu$, $T_\alpha^1 = \{\langle g_i : i < \alpha \rangle : g_i$ is a function from i into $i\}$.

We can look at $\langle g_i : i < \alpha \rangle$ as a sequence of approximations to the desired f. We call j a failure of $\bar{g} = \langle g_i : i < \alpha \rangle$ if

(α) $j < \alpha$.

(β) (A) or (B) where

(A) for arbitrarily large $\gamma < j$, $\langle g_i(\gamma) : \gamma < i < j \rangle$ is not eventually $g_j(\gamma)$.

(B) there is no $\beta < \mu$ satisfying:

(i) $f_\beta \restriction j$, g_j are eventually equal (i.e.,

$$(\exists \xi < j)(\forall i)[\xi \leq i < j \to f_\beta(i) = g_j(i)]) \quad \text{and}$$

(ii) f_β maps bounded subsets of j to bounded subset of j (i.e.,

$$(\forall \xi < j)(\exists \zeta < j)[(\forall i < \xi) f_\beta(i) < \zeta]).$$

(γ) j is a strong limit cardinal.

Let, for $\alpha < \mu$, $T_\alpha = \{(\bar{g}, h) : \bar{g} \in T_\alpha^1, h$ is one-to-one, regressive (i.e., $h(j) < j$) and Dom $h = \{j : j$ is a failure of $\bar{g}\}\}$.

Naturally $(\bar{g}, h) \leq (\bar{g}_\alpha^1, h^1)$ if $\bar{g}$ is an initial segment of $\bar{g}^1$ and $h = h^1 \restriction \mathrm{Dom}\, h$, so $T = \bigcup_{\alpha<u} T_\alpha$ is a tree with T_α its αth level.

Note that if $\bar{g}^1 = \bar{g} \restriction \alpha$ and $j < \alpha$ then j is a failure of $\bar{g}$ iff it is a failure of $\bar{g}^1$. It clearly is sufficient to prove the following two assertions:

Assertion 1. If T has a μ-branch, the required f exists.

Let $(\bar{g}^\alpha, h^\alpha) \in T$ $(\alpha < \mu)$ be the member of the branch in level α, so $\bar{g}^\alpha = \langle g_i: i < \alpha\rangle$ and let $\bar{g} = \langle g_i: i < \mu\rangle$, $h = \bigcup_{\alpha<\mu} h_\alpha$, so clearly Dom h is the set of failures of $\bar{g}$, and h is one-to-one. As h is one-to-one, for some closed and unbounded $S \subseteq \mu$ for every $\alpha \in S$ and $j \in \text{Dom } h$, $j < \alpha$ iff $h(j) < \alpha$, and w.l.o.g. every $\alpha \in S$ is a strong limit cardinal. So each $\alpha \in S$ does not belong to Dom h (as then $h(\alpha) < \alpha$) hence is not a failure of $\bar{g}$. So (by condition β(A)) there is $\gamma_0(\alpha) < \alpha$ (for $\alpha \in S$) such that $\gamma_0(\alpha) \le \gamma < \alpha$ implies $\langle g_i(\gamma): i < \alpha\rangle$ is eventually $g_\alpha(\gamma)$ and there are (by β(B)) ordinals $\beta(\alpha)$, $\gamma_1(\alpha)$, $\gamma_2(\alpha)$ such that: $\gamma_1(\alpha) \le \gamma < \alpha \Rightarrow f_{\beta(\alpha)}(\gamma) = g_\alpha(\gamma)$ and $\gamma < \gamma_1(\alpha) \Rightarrow f_{\beta(\alpha)}(\gamma) < \gamma_2(\alpha)$, and $\gamma_1(\alpha), \gamma_2(\alpha) < \alpha$, $\gamma_0(\alpha) \le \gamma_1(\alpha)$.

By Fodour's theorem, as $\gamma_l(\alpha) < \alpha$, for some stationary set $S_1 \subseteq S$ for every $\alpha \in S_1$, $\gamma_l(\alpha) = \gamma_l$ $(l = 0, 1, 2)$, and (by β(B)(i)) for some stationary set $S_2 \subseteq S_1$, $f_{\beta(\alpha)} \restriction \gamma_1$ is equal for every $\alpha \in S_2$. So we can define a function g by: for $\gamma_0 \le \gamma < \mu$, $g(\gamma)$ is the eventual value of $\langle g_i(\gamma): i < \mu\rangle$ (exists as $\langle g_i(\gamma): i < \alpha\rangle$ is eventually constant for every $\alpha \in S_2$, so for some $i^0(\alpha) < \alpha$, $g_i(\gamma) = g_{i^0(\alpha)}(\gamma)$ for every i, $i^0(\alpha) \le i < \alpha$, so for some stationary $S_3^\gamma \subseteq S_2$, $\alpha \in S_3^\gamma \Rightarrow i^0(\alpha) = i^0$ so $g_i(\gamma) = g_{i^0}(\gamma)$ for every $i \ge i^0$). Now for $\gamma < \gamma_1$, $g(\gamma)$ is $f_{\beta(\alpha)}(\gamma)$ for every $\alpha \in S_2$. So g is as required.

Assertion 2. T is a μ-tree. As clearly the number of elements in level α is $< \mu$ (in fact $\le 2^{|\alpha|}$), we have to prove that there are elements in level α for each $\alpha < \mu$.

For levels before the first strong limit it is trivial, so it suffices to prove:

(*) if $(\bar{g}, h) \in T_{\alpha+1}$, $\bar{g} = \langle g_i: i \le \alpha\rangle$, $\alpha < \beta$ *then* there is $(\bar{g}^1, h^1) \in T_\beta$ such that

(A) $i \in \text{Dom } h^1$, $\alpha < i < \beta$ implies $h^1(i) \ge \alpha$.

(B) $(\bar{g}, h) \le (\bar{g}^1, h^1)$.

(C) $\alpha < i < \beta$ implies $g_i^1 \restriction \alpha = g_\alpha$.

We prove (*) by induction on β, and let $\beta = \delta + n$, $n < \omega$, δ limit.

Case (a). $\delta < \beth_\omega$.

No problems as there are no failures.

Case (b). δ not strong limit.

Choose $\gamma + 1 < \delta$, such that there is no strong limit σ, $\gamma \le \sigma \le \delta$. By the induction hypothesis there is $(\bar{g}^1, h^1) \in T_{\gamma+1}$, $(\bar{g}, h) \le (\bar{g}^1, h^1)$ as in (*). Now choose any $\bar{g}^2 \in T_\beta^1$ such that $\bar{g}^2 \restriction (\gamma + 1) = \bar{g}^1$ and $(\bar{g}^2, h^1)$ satisfies (B) and (C). Then $(\bar{g}^2, h^1)$ is as required.

Case (c). δ strong limit singular.

Choose $\gamma < \delta$, $\gamma > \alpha$, $\gamma > \text{cf } \delta$, and let $\delta = \sum_{\eta<\text{cf }\delta} \sigma(\eta)$, $\sigma(\eta)$ $(\eta < \text{cf } \delta)$ increasing and continuous, $\sigma(0) = \alpha + 1$, $\sigma(1) > \gamma$, $\sigma(2) > \sigma(1) + \gamma$, but no σ, $\sigma(1) \le \sigma \le \sigma(2)$ is strong limit. Let $\sigma(\text{cf } \delta) = \delta$. Now we define by induction on $\eta \le \text{cf } \delta$, $(\bar{g}^\eta, h^\eta) \in T_{\sigma(\eta)+1}$, increasing in the tree so that Range h^η is disjoint to $\{i: \sigma(1) + \eta < i < \sigma(2)\}$. Then it is easy to prove (*).

For $\eta = 0$. $(\bar{g}^0, h^0) = (\bar{g}, h)$.

For η a successor ordinal $\neq 2$. Use the induction hypothesis.

For $\eta = 2$. Use Case (b), hence Dom $h^2 \subseteq \sigma(1)$.

For η limit. In $\bar{g}^\eta = \langle g_i^\eta: i \le \sigma(\eta)\rangle$ we already determine g_i^η for $i < \sigma(\eta)$ and $g_{\sigma(\eta)}^\eta$ will be any appropriate function extending g^α.

As for h^η, we already essentially define $h^\eta \restriction \sigma(\eta)$, and if we have to define $h^\eta(\sigma(\eta))$, we define it as $\sigma(1) + \eta$ (by a demand on the construction, h^η is one-to-one).

Case (d). δ is strong limit and inaccessible.

There is a closed unbounded set $S \subseteq \delta$ such that for $\sigma \in S$, $\eta < \sigma \Rightarrow f_\delta(\eta) < \sigma$. Let $\langle \sigma(\eta) : \eta < \delta \rangle$ be an enumeration of the limit points of S ($\sigma(\eta)$ increasing and continuous) and $\alpha < \sigma(0)$.

The proof is essentially like Case (c). We define by induction on η $(\bar{g}^\eta, h^\eta)$, increasing in the tree $\bar{g}^\eta = \langle g_i^* : i \leq \sigma(\eta) + 1 \rangle$ such that for $i > \sigma(\eta), \alpha \leq \gamma < \sigma(\eta)$, $g_i^*(\gamma) = g_\delta(\gamma)$.

For η a successor. By the induction hypothesis we can find $(\bar{g}_1^\eta, h^\eta) \geq (\bar{g}^{\eta-1}, h^{\eta-1})$ in $T_{\sigma(\eta)+1}$ as in (*). Now define g_i^* for $i = \sigma(\eta) + 1$ by:

$$g_i^*(j) = \begin{cases} g_\alpha(j) & \text{when } j < \alpha, \\ f_\delta(j) & \text{when } \alpha \leq j < \sigma(\eta), \\ 0 & \text{when } j = \alpha(\eta). \end{cases}$$

Let $\bar{g}_1^\eta = \langle g_i^* : i < \sigma(\eta) + 1 \rangle$, so $\bar{g}^\eta$ is defined.

For η limit. We define g_i^* for $i = \sigma(\eta), \sigma(\eta) + 1$ as above and no failure occurs in $\sigma(\eta)$.

Now it is easy to define the required $(\bar{g}^1, h^1) \in T_\beta$.

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