This is the exact solution of exercise 4.32 of the book. The solution provided in the book neglects that two events can happen within one day. This is an unnecessary assumption that introduces considerable error unless the transition rates are low (i.e., unless most likely nothing happens within a day). It is of general interest to know how to switch from a continuous-time system to a discrete-time population census. Further, the book's solution assumes that patients are discharged only once a day rather than continuously (this is realistic but not mentioned in the exercise), and takes $1/\delta$ to be the probability of discharge at the end of the day. This is seriously misleading given that in the main text, $1/\delta$ is the *rate* of discharge (which, unlike a probability, has the unit 1/time, making an expression like $1 - 1/\delta$ meaningless).

The aim of the exercise is to find the probability P_{ij} that the ICU is in state i tomorrow if it is in state j today. Let us assume that patients are discharged and new patients admitted at noon each day, and the state of the ICU is taken just after discharge. Then first we have one day of continuous dynamics ("within-day dynamics") given by equations 4.46 of the book but without discharge, i.e., by

$$\begin{aligned} \frac{dp_0}{dt} &= -2\alpha p_0 \\ \frac{dp_1}{dt} &= 2\alpha p_0 - (\alpha+\beta)p_1 \\ \frac{dp_2}{dt} &= (\alpha+\beta)p_1 \end{aligned} \tag{1}$$

Notice that without discharge during the day, the number of infected patients can only increase, i.e., the ICU can go from state 0 to 1 to 2 but not backwards. The first two equations are therefore autonomous, and since $p_2(t) = 1 - p_0(t) - p_1(t)$, we don't need to use the last equation to know the full system.

Let us start with an ICU in state 0, i.e., with the initial condition $p_0(0) = 1$, $p_1(0) = p_2(0) = 0$ (this is the most painful technically, the rest will be easy). The first equation of (1) integrates to $p_0(t) = e^{-2\alpha t}$. Substituting this into the second equation we get the linear inhomogeneous ODE

$$\frac{dp_1}{dt} = 2\alpha e^{-2\alpha t} - (\alpha+\beta)p_1$$

the solution of which is $p_1(t) = f(t)$ with

$$f(t) = 2\alpha e^{-\alpha t} \, \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}$$

(there is a removable discontinuity at $\beta = \alpha$). At the end of the day ($t = 1$), but still before discharge, the ICU is in state 0 with probability $e^{-2\alpha}$, in state 1 with probability $f(1)$, and in state 2 with probability $1 - e^{-2\alpha} - f(1)$.

Starting from state 1 (i.e., $p_1(0) = 1$, $p_0(0) = p_2(0) = 0$), the first equation of (1) yields $p_0(t) = 0$; the ICU cannot get into state 0 before the next discharge. From the second equation of (1), the ICU remains in state 1 by the end of the day with probability $e^{-(\alpha+\beta)}$, and gets to state 2 with probability $1 - e^{-(\alpha+\beta)}$. Finally, if the ICU starts from state 2 (i.e., $p_2(0) = 1$, $p_0(0) = p_1(0) = 0$), it will remain in state 2 with probability 1.

We collect the above results into the matrix

$$\mathbf{W} = \begin{bmatrix} e^{-2\alpha} & 0 & 0 \\ f(1) & e^{-(\alpha+\beta)} & 0 \\ 1 - e^{-2\alpha} - f(1) & 1 - e^{-(\alpha+\beta)} & 1 \end{bmatrix}$$

where W_{ij} (indexing the rows/columns 0,1,2) is the probability that the ICU is in state i at the end of the within-day dynamics if it started in state j (the matrix is lower triangular because during the day, the number of infected patients cannot decrease). Next, we consider how the state of the ICU changes by discharge. Each of the two patients is discharged with probability q independently of each other, and each discharged patient is replaced with a newly admitted uninfected patient. Let Q_{ij} denote the probability that the ICU is in state i after discharge if it was in state j before discharge; these probabilities are the elements of the matrix

$$\mathbf{Q} = \begin{bmatrix} 1 & q & q^2 \\ 0 & 1-q & 2q(1-q) \\ 0 & 0 & (1-q)^2 \end{bmatrix}$$

(notice that this matrix is upper triangular because by discharge, the number of infected patients cannot increase). Finally, we obtain the full day's dynamics by first applying the matrix $\mathbf{W}$ to the initial vector and then applying $\mathbf{Q}$. Hence the day-to-day projection matrix is $\mathbf{P} = \mathbf{QW}$.

The elements of $\mathbf{P}$ are readily interpretable. For example, for the probability that from state 2 the ICU gets to state 0 we obtain $P_{02} = q^2$; this is because starting from state 2, the ICU will remain in state 2 during the day, and then both patients are replaced with uninfected patients with probability q^2. A more complicated element is $P_{00} = e^{-2\alpha} + qf(1) + q^2(1 - e^{-2\alpha} - f(1))$. Starting from state 0, the ICU may get to state 0 in three different ways: both patients remain uninfected, in which case it does not matter whether they are replaced at the end of the day (first term); one patient gets infected but the other not, and the infected patient is replaced (second term); or both patients get infected and both are replaced (third term).