Dependence Logic
Exercise 10

1. Let ψ be of the form $[\mathrm{LFP}_{\vec{x},R}\,\phi]\vec{t}$, where ϕ is a first-order formula. Show that

$$\psi \equiv \theta,$$

where $\theta \in \Pi^1_1$. Here Π^1_1 is a fragment of second-order logic consisting of formulas of the form form

$$\forall Q_1 \cdots \forall Q_n \chi,$$

where $Q_1, ..., Q_n$ are function and relation variables and χ is a first-order formula.

2. Let ψ be of the form $[\mathrm{LFP}_{\vec{x},R}\,\phi(\vec{x}, y)]\vec{x}$, where ϕ is a first-order formula with free variables $\vec{x}$ and y. Show that ψ is equivalent with the formula

$$[\mathrm{LFP}_{\vec{x}y,R'}\,\phi(R'(\vec{t}y)/R(\vec{t}))]\vec{x}y.$$

3. Complete the proof of a theorem from Monday's lecture: Show that

$$\mathcal{M} \models_X \phi \Leftrightarrow (\mathcal{M}, X(\vec{x})) \models_s \phi^*(R, \vec{x}) \text{ for all } s \in X,$$

for $\phi := \forall v \psi(\vec{x}, v)$ and

$$\phi^*(R, \vec{x}) := \forall v[\mathrm{GFP}_{\vec{x}v,S}\,(R(\vec{x}) \wedge \psi^*(S, \vec{x}v)]\vec{x}v.$$