PROBLEM SET 2 SOLUTIONS Q 4 AND Q 5

Take κ to be a measurable cardinal, $\mathcal{U}$ an ultrafilter witnessing this, M the transitive collapse of $\langle V/\mathcal{U}, E_{\mathcal{U}}\rangle$ and $j : V \to_e M$ the associated elementary embedding. We have:
(4) ${}^{\kappa}M \subseteq M$
(5) $2^\kappa \leq (2^\kappa)^M < j(\kappa) < (2^\kappa)^+$

Proof. (4) Let $\langle a_\alpha | \alpha \in \kappa\rangle$ be a sequence with each $a_\alpha \in M$. Take g_α, h functions on κ such that $[g_\alpha] = a_\alpha$ and $[h] = \kappa$. We construct F such that $[F] = \langle a_\alpha | \alpha \in \kappa\rangle$: Let $F(\beta) = \langle g_\alpha(\beta) | \alpha < h(\beta)\rangle$[1] for $\beta < \kappa$. We now use Łos' Theorem repeatedly. As $F(\beta)$ is a sequence of length $h(\beta)$ for all $\beta \in \kappa$, we have that $[F]$ is a sequence of length $[h] = \kappa$. Also for $\alpha < \kappa$ we have $\{\beta : \alpha < h(\beta)\} \in \mathcal{U}$ so $\{\alpha : \text{the } \beta^{th} \text{ place of } F(\alpha) = g_\beta(\alpha)\} \in \mathcal{U}$. Thus the β^{th} place of $[F] = [g_\beta] = a_\beta$, so F is as intended.

(5) From the fact that $\forall \alpha < \kappa(\alpha \in M)$ and part (i) it follows that $P^M(\kappa) = P(\kappa)$. As $M \subseteq V$ we see $2^\kappa \leq (2^\kappa)^M$. As κ is inaccessible in V it follows by elementarily that $j(\kappa)$ is inaccessible in M so we must have $(2^\kappa)^M < j(\kappa)$. Let $\alpha \in j(k)$ and $[f] = \alpha$. Then by Łos Theorem we have $\{\beta < \kappa : f(\beta) < \kappa\} \in \mathcal{U}$ so without loss of generality we may assume $f \in {}^{\kappa}\kappa$. Thus $|j(\kappa)| \leq \kappa^\kappa = 2^\kappa$ so $j(\kappa) \leq (2^\kappa)^+$ (and $j(\kappa)$ is not a cardinal).

□

[1] There is a mistake in Jech here - this is the corrected definition.