Topology II

Exercise 11 (30.11.2017)

1. Suppose, that $\mathcal{T}$ and $\mathcal{T}'$ are topologies in a set X and $\mathcal{T} \subset \mathcal{T}'$. Prove, that if $(X, \mathcal{T}')$ is separable, then also $(X, \mathcal{T})$ is separable.

2. Prove, that

a) a discrete subspace of an N_2 space is countable.

b) the space $\mathbb{R}^{\mathbb{R}}$ has an uncountable discrete subspace.

3. (Propositions 13.19 and 13.20) Prove, that components are always closed subsets. Prove, that if the number of components is finite, then they are also open.

4. a) Suppose, that E_j, $j \in J$, are connected subsets of a space X, which have a common element. Prove, that the union of the sets E_j is connected.

b) Suppose, that $X = \bigcup\{A_j \mid j \in \mathbb{N}\}$, where the sets A_j are connected and $A_j \cap A_{j+1} \neq \emptyset$ for all $j \in \mathbb{N}$. Prove, that X is connected. Hint: Prove first by induction, that $A_1 \cup \cdots \cup A_n$ is connected for all $n \in \mathbb{N}$.

5. Let

$$A = \{(x, \sin \frac{1}{x}) \mid 0 < x \leq 1\} \subset \mathbb{R}^2.$$

Prove, that the closure $\overline{A}$ of A is not path connected.

6. Watch the video "Who cares about topology? (Inscribed rectangle problem)" at https://www.youtube.com/watch?v=AmgkSdhK4K8

There is a connection with the solution, and a fact mentioned (not proved, just mentioned) previously at our course. Which fact? (No exact proofs are needed here.)

[Hint: Consider the closed disc $\overline{B}^2$, its' boundary S^1 and the Möbius strip M. The boundary circle ∂M of M is homeomorphic with S^1, let $f\colon S^1 \to \partial M$ be a homeomorphism. It can be proved that if you glue the disc to the Möbius strip using the map f, then the adjunction space is homeomorphic with the projective plane.]