Dependence Logic
Exercise 11

1. Define propositional connectives $\varovee$ and $\rightarrow$ (intuitionistic disjunction and implication) as follows:

$$X \models \phi \varovee \psi \Leftrightarrow X \models \phi \text{ or } X \models \psi$$

$$X \models \phi \rightarrow \psi \Leftrightarrow \forall Y \subseteq X : Y \models \phi \Rightarrow Y \models \psi.$$

Show that the extension $PL(\varovee, \rightarrow)$ of PL by these connectives is a downwards closed logic (see Proposition 84).

2. Show that dependence atoms $=(\vec{p}, q)$, where $\vec{p} = (p_1, \ldots, p_n)$ can be expressed in $PL(\varovee)$.

3. Show using induction on $\phi \in PL(\perp_c)$ that if $X \models \phi$ holds then $\{s\} \models \phi$, for all $s \in X$.

4. Show that there are sets of teams F with domain $\{p_1, p_2\}$ such that there is no formula $\phi \in PL(\perp_c)$ such that

$$F = \{X \mid X \models \phi,\ \mathrm{Dom}(X) = \{p_1, p_2\}\}.$$

In other words, $PL(\perp_c)$ is not an expressively complete propositional logic.

5. Define a propositional connective $\triangledown$ (nonemptiness operator) as follows:

$$X \models \triangledown\phi \Leftrightarrow X = \emptyset \text{ or there exists } Y \subseteq X \text{ s.t. } Y \neq \emptyset \text{ and } Y \models \phi.$$

Show that the formulas of $PL(\triangledown)$ have the closure under unions property (Proposition 85).