Additional exercises

1. *Treated vs untreated infectives.* Suppose an infection starts with a latent phase, from which the individuals proceed to the untreated infectious state at a rate ν. Untreated infectives are detected by the health care system at a rate γ, upon which they shall receive treatment. While untreated, infectives recover at a rate α_U, whereas when receiving treatment, they recover at a rate α_T. Contact occurs according to frequency-dependent transmission. Untreated infectives have a contact rate c_U and upon contact, transmit the disease with probability p_U; the same parameters for treated infectives are c_T and p_T, respectively. All individuals hav the same death rate, μ.

(a) Write down the ODEs for the linearized dynamics valid for the beginning of the outbreak.
(b) Construct the transmission matrix and the inverse transition matrix.
(c) Obtain the next generation matrix with large domain.
(d) Determine R_0.

2. *Wolbachia infections.* The bacterium *Wolbachia* is an intracellular parasite of many insects. It can spread by horizontal transmission, but its most important way of spreading is vertically through the eggs of infected females: a fraction w (close to 1) of the eggs of infected females carry the bacterium so that the offspring (both sons and daughters) develop infected. *Wolbachia*, however, cannot spread through the sperm, so that males are dead ends for vertical transmission. With a clever trick, *Wolbachia* can manipulate the sex ratio of the offspring of infected females, so that they produce more daughters than sons (i.e., more infected females who will further spread *Wolbachia* instead of dead-end infected males).

Suppose that horizontal transmission occurs with mass action, with males and females being equally infectious. Infected females produce daughters and sons in proportions $q : 1 - q$, and a fraction w of the offspring are infected. Infected individuals die at a rate α, recovery is not possible. In the uninfected population, the sex ratio is 1:1.

(a) Write down the linearized dynamics.
(b) Construct the next generation matrix.

(c) Derive the equations that determine the probability of a major outbreak if it is started with (i) one infected male; (ii) one infected female.

3. *Viral infection of different tissues.* Here we consider the dynamics of a viral infection within one host individual. Suppose the virus can infect two different tissues of n_1 and n_2 cells, respectively. The virus particles attach to the cells according to mass action with the tissue-specific rates β_1 and β_2. The infected cells produce new virus particles at rates γ_1 and γ_2, and they die at rates δ_1 and δ_2. The free viruses decay at rates μ_1 and μ_2. The new viruses are released within the tissue where they were produced. The two tissues, however, exchange free viruses through the blood stream; a free virus gets from tissue j to tissue i ($i \neq j$) with a rate m_{ij}.

(a) Construct the ODEs of the linearized dynamics.
(b) Let us define a generation from a free virus to a free virus, starting at the point where a new virus is released by an infected cell. Give the elements of the next generation matrix.

4. *Chicken pox.* Chicken pox is highly infectious before the symptoms appear, but less so after the symptoms (partly because it is easy to recognise and isolate the victims). Suppose that a fraction q of the hosts are partially defended such that even though they get the disease, they do not become very infectious and recover faster; the remaining fraction $1 - q$ is undefended. A type i host ($i = 1, 2$) transmits the disease at a rate β_{Bi} before the symptoms appear and at a rate β_{Ai} after. The symptoms appear after an exponentially distributed waiting time, which has parameter ν_i. After the symptoms, recovery occurs at a rate α_i. The contact rate is constant c (not mass action; if the contact rate decreases after the symptoms, we factor the difference into β_{Ai}).

(a) Construct the next generation matrix and show that it is of rank 1.
(b) Give R_0.
(c) Generalize the model assuming that there is a continuous distribution of hosts with a probability density function $q(s)$ of types and type-dependent rates $\beta_B(s), \beta_A(s), \nu(s), \alpha(s)$. Write down the next generation operator and determine R_0.