Topology II

Exercise 12 (7.12.2017)

1. Prove the Invariance of Domain theorem 14.3 in the case $n = 1$.

2. a) Prove, that every manifold is locally connected.

b) Prove, that a manifold is path connected, if and only if it is connected.

3. Prove, that every manifold is normal.

4. Is the interval $[0, 2]$ compact in the topology $\mathcal{T}_{pa}$?

5. Suppose, that $(X, \mathcal{T})$ is compact, $(X, \mathcal{T}_1)$ is Hausdorff, and $\mathcal{T}_1 \subset \mathcal{T}$. Prove, that $\mathcal{T}_1 = \mathcal{T}$. [Hint: Apply Proposition 15.18 to the map $\mathrm{id}\colon (X, \mathcal{T}) \to (X, \mathcal{T}_1)$.]
This result states, that a compact Hausdorff topology in a space is a minimal Hausdorff topology, that is, there doesn't exist a Hausdorff topology, which is strictly coarser.

6. Let $n \geq 2$ and $X = \mathbb{R}^n \setminus \{\bar{0}\}$.

a) Prove, that the set $X \setminus S^{n-1}$ has exactly two components U and V (which?).

b) Suppose, that $f\colon X \approx \mathbb{R}^n$. Prove, that one of the sets fU and fV is bounded, and thus its' closure in $\mathbb{R}^n$ is compact.

c) Prove, that this leads to a contradiction, and thus $X \not\approx \mathbb{R}^n$.

[Hint: You may need facts from Topology I concerning connectedness and compactness.]