A comment on section 8.3, pp. 220-221, the probability of a minor outbreak

The distribution of new infections, $m(x, \omega)$, is generally not independent of k, the number of new infections an individual makes. For (8.25), $m(x, \omega)$ has to be conditioned on k.

As a simple example for the lack of independence, suppose that ω_1, ω_2 are two disjunct subsets of Ω. Before a certain infection age τ_0, infecteds produce new infections only among susceptibles with $x \in \omega_1$; after age τ_0, only with $x \in \omega_2$. The infecteds recover after an exponentially distributed time. In this case, those who happen to recover early will have low k and all new cases they made are in ω_1; those who recover late have higher k and new cases both in ω_1 and ω_2. Hence the distribution of new cases is not independent of k.

In this example, the distribution of new infections and k are *conditionally* independent. This means if we condition on recovery time T, i.e., we consider only those infecteds who recover at an infection age $\tau \in (T, T + dt)$, then within this group, independence holds (provided there is no other difference between infecteds). We can therefore include the recovery time in the state-at-infection η to regain the formulation in section 8.3.

In general, one has to be aware that while $A(\tau, x, \eta)$ can be an expectation averaging over individual variation in deterministic models (R_0, final size), for the probability of a minor outbreak, individual variability in $A(\tau, x, \eta)$ matters. For example, in the SIR-model (which has only one state-at-infection and hence A does not depend on x and η), the expectation $A(\tau) = \beta e^{-\alpha\tau}$ averages over individuals with different (exponentially distributed) recovery times. The infectivity of a single individual is however a step-function, with $A(\tau)$ taking value β before the individual's recovery and 0 after.

Equation (8.25) holds if the infecteds individually have the same infectivity function $(\tau, x, \eta) \mapsto A(\tau, x, \eta)$. If we have variability, as the variable recovery time in the example above, we can include this variability as a component in η, so that individuals with the same η have the same $A(\tau, x, \eta)$ for each value of τ and x. With $A(\tau, x, \eta)$ being the same for each individual, it is the same also for those having different k, i.e., we have $A(\tau, x, \eta | k) = A(\tau, x, \eta)$ and therefore $m(x, \omega | k)$ is the same as $m(x, \omega)$.

Individual variability in recovery time can be admitted in (8.25) in two special cases. First, if the pattern of transmission (which η infects which x) is independent of the age of the infection such that $A(\tau, x, \eta)$ factors as $A(\tau, x, \eta) = g(\tau)h(x, \eta)$, then the integral of $g(\tau)$ cancels in (8.27) and we can admit variability in the length of the infectious period. In the example above, this means that ω_1- and ω_2-individuals would be infected simultaneously. This may be a common pattern in real life. Second, (8.25) holds if each individual has the same value for the integral $\int_0^\infty A(\tau, x, \eta) d\tau$; this means that if someone recovers faster, then he has proportionally higher rates of making new infections while infectious. This seems unlikely, especially when the time of recovery is random.

Note that there may also be individual variability in the pattern of infection (which η infects which x), and this variability also matters for the probability of minor outbreak. Continuing with the example above, if some individuals with state-at-infection η tend to infect ω_1-susceptibles whereas others with the same value of η tend to infect ω_2-susceptibles, then η is not a sufficient characterisation of the state-at-infection; the tendency to infect different susceptibles needs to be incorporated into η (e.g. as a new element of a vector-valued state at infection).

For the role of individual variability on the probabilility of a minor outbreak, recall the end of section 2.1 (pp. 38-39), and the two specific examples in equations (1.7) and (1.8) (for fixed recovery time (1.7) is obtained in the main text, for exponentially distributed recovery time (1.8) is derived in exercise 1.13 on p 9, and the two are compared in Figure 3.3).