Topology II

Exercise 13 (14.12.2017)

1. Let X be a space, and (x_n) a sequence in X, which converges to $a \in X$. Prove, that the set $A = \{a\} \cup \{x_n \mid n \in \mathbb{N}\}$ is compact.

2. A continuous map $f\colon X \to Y$ is called *proper*, if the inverse image of each compact set of Y is compact. Which of the following maps $f\colon \mathbb{R} \to \mathbb{R}$ are proper: (a) $f(x) = \sin x$, (b) $f(x) = x^2$, (c) $f(x) = 2$.

3. Is the set $W = \{f_n \mid n \in \mathbb{N}\}$ of maps $f_n\colon \mathbb{R} \to \mathbb{R}$ equicontinuous, if
(a) $f_n(x) = x^2/n$,
(b) $f_n(x) = (x+n)^2$,
(c) $f_n(x) = 1/(1+nx^2)$?

4. Suppose, that X is locally compact, Hausdorff and N_2. Prove, that there exists a sequence of open sets $U_1, U_2, \ldots$, such that $\overline{U}_j$ is compact, $\overline{U}_j \subset U_{j+1}$ and X is the union of the sets U_j.

5. Let $X \neq \emptyset$ be a compact Hausdorff space and $f\colon X \to X$ continuous. Prove, that there exists a closed subset $A \subset X$, such that $A \neq \emptyset$ and $fA = A$. [Hint: Consider the sets $fX, ffX = f^2X, \ldots$]

6. Suppose that X is a compact Hausdorff space, and (f_n) an increasing sequence of continuous functions $f_n\colon X \to \mathbb{R}$, which converges pointwise to a continuous function $g\colon X \to \mathbb{R}$. Prove, that the convergence is uniform in X. Prove by giving an example, that the result doesn't hold without the assumption, that the sequence is increasing. [Hint: Let $\varepsilon > 0$. Denote $A_n = \{x \in X \mid f_n(x) \leq g(x) - \varepsilon\}$ and use Proposition 15.20.]