Dependence Logic
Exercise 12

1. Define propositional connective $\otimes$ as follows:

$$X \models \varphi \otimes \psi \quad \iff \quad \forall Y, Z \subseteq X: \text{ if } Y \cup Z = X, \text{ then } Y \models \varphi \text{ or } Z \models \psi.$$

Show that $\otimes$ is the dual of $\vee$, that is

$$\varphi \otimes \psi \equiv \sim (\sim \varphi \vee \sim \psi).$$

2. Show using $\otimes$ and the previous exercise that the intuitionistic implication (see Definition 88) can be expressed in PL($\sim$).

3. Define an atom $max(q_1, \ldots, q_n)$ as follows:

$$X \models max(q_1, \ldots, q_n) \quad \iff \quad \{(s(q_1), \ldots, s(q_n)) \mid s \in X\} = \{0, 1\}^n.$$

Show that $max(q_1, \ldots, q_n)$ can be defined in PL($\sim$) by the formula

$$\sim \bigvee_{i=1}^{n} {=}(q_i).$$

4. Show that the exchange rule of independence atoms is sound, that is, show that for all $\mathcal{M}$ and X, if $\mathcal{M} \models_X \vec{x} \perp \vec{y}$ and $\mathcal{M} \models_X \vec{x}\vec{y} \perp \vec{z}$, then $\mathcal{M} \models_X \vec{x} \perp \vec{y}\vec{z}$.

5. Show that the constancy rule of independence atoms is sound, that is, show that for all $\mathcal{M}$ and X, if $\mathcal{M} \models_X \vec{x} \perp \vec{x}$ and $\mathcal{M} \models_X \vec{y} \perp \vec{z}$, then $\mathcal{M} \models_X \vec{x}\vec{y} \perp \vec{z}$.