HOMEWORK 2 (EC and MF)

(1) (10pts) Consider $V \subset \mathbb{P}^2(\mathbb{C})$ defined by $y^2z = x^3 + z^3$. Show that the map $f : V \to \mathbb{P}^2(\mathbb{C})$ where $f = [x^2, xy, z^2]$ is a *morphism.*

(2) (10pts) Suppose $x^2 + y^2 = 7z^2$ where $x, y, z \in \mathbb{Q}$, then $x = y = z = 0$. (hint: see Example 2.5 of [Sil])

(In case you are interested: if you replace 7 with 11, 19, 23, or any *prime* number $p \equiv 3 \pmod 4$, same thing happens; this phenomenon is related with *quadratic residue*).

(3) (15pts) Consider $V \subset \mathbb{P}^2(\mathbb{C})$ defined by $x^2 + y^2 = 5z^2$.

(a) Suppose $A^2 + B^2 = 5$ with $A, B \in \mathbb{C}$. Let $a = A - 1, b = B - 2$, calculate the equation satisfied by a, b. Suppose $\lambda = \frac{a}{b}$, then express both a, b in terms of λ.

(b) Use (1) to find ALL $\mathbb{Q}$-solutions of $A^2 + B^2 = 5$.

(c) Show that $V(\mathbb{Q})$ is isomorphic to $\mathbb{P}(\mathbb{Q})$.

(In case you are interested: if you replace 5 with 13, 17, or any *prime* number $p \equiv 1 \pmod 4$, same thing happens)