Exercise 4
Consider the evolution of a virus in a population with dynamics given by
\[
\begin{align*}
\frac{dS}{dt} &= \lambda(N)S - \mu S - S \sum_{j=1}^{k} \beta(x_j)I_j \\
\frac{dI_i}{dt} &= S\beta(x_i)I_i - \mu I_i - \alpha(x_i)I_i
\end{align*}
\]
where \( S \) is the density of susceptible (i.e., non-infected) individuals, \( I_i \) the density of individuals infected by virus type \( x_i \) for \( i = 1, \ldots, k \), and where \( N = S + \sum_{j=1}^{k} I_j \) is the total population density, and where \( \lambda, \mu, \alpha \) and \( \beta \) are all positive parameters.

(a) If you can, interpret the structure of the model and the meaning of the parameters.

(b) Rewrite the above system in the form of the resident dynamics type 2 as introduced in the lecture of 29-01-2018.

(c) Write down the invader dynamics for the strategy \( y \) and corresponding population density \( m \), and give an expression for the invasion fitness.

(d) Use the principle of selective neutrality of residents to show that coexistence of different resident strategies is not possible except under very special conditions. What are these special conditions?

(e) Show that the invasion fitness in a monomorphic resident population with strategy \( x \) and corresponding population density \( n \) is positive if and only if
\[
\frac{\alpha(y) + \mu}{\beta(y)} < \langle S(t) \rangle
\]
where \( \langle S(t) \rangle \) is the longterm time average of \( S(t) \).

(f) Use the principle of selective neutrality of residents to show that \( (\alpha(x) + \mu)/\beta(x) \) in successive residents decreases with every invasion-substitution event. Show that this means that the evolution of the virus in the long run minimizes the time-average of the population density of the susceptible individuals.
Exercise 5:
As a variation on the previous exercise, consider the system
\[
\begin{align*}
\frac{dS}{dt} &= \lambda S - \mu(N)S - S \sum_{j=1}^{k} \beta(x_j)I_j \\
\frac{dI_i}{dt} &= S\beta(x_i)I_i - \mu(N)I_i - \alpha(x_i)I_i
\end{align*}
\]
where now the \( \mu \) is a function of the total population density instead of the \( \lambda \).

(a) If you can, interpret the structure of the model and the meaning of the parameters.

(b) Rewrite the above system in the form of the resident dynamics type 2 as introduced in the lecture of 29-01-2018.

(c) Write down the invader dynamics for the strategy \( y \) and corresponding population density \( m \), and give an expression for the invasion fitness.

(d) Use the principle of selective neutrality of residents to show that coexistence of two different resident strategies cannot be excluded, but that coexistence of more than two different strategies is not possible except under very special conditions. What are these special conditions?

(e) Use the principle of selective neutrality of residents to find an explicit expression of environment in a dimorphic resident population.

Exercise 6
Consider the resident dynamics for different prey strategies \( x_1, \ldots, x_k \) with corresponding population densities \( n_1, \ldots, n_k \) and a single non-evolving predator with population density \( p \) given by the system
\[
\begin{align*}
\frac{dn_i}{dt} &= r(x_i) n_i \left( 1 - \sum_{j=1}^{k} n_j \right) - \frac{\alpha(x_i)n_i p}{1 + \sum_{j=1}^{k} \beta(x_j)n_j} \\
\frac{dp}{dt} &= p \left( \frac{\sum_{j=1}^{k} \gamma(x_j)n_j}{1 + \sum_{j=1}^{k} \beta(x_j)N_j} - \delta \right)
\end{align*}
\]
for positive \( r, \alpha, \beta, \gamma \) and \( \delta \).

(a) If you can, interpret the structure of the model and the meaning of the parameters.

(b) Rewrite the above system in the form of the resident dynamics type 2 as introduced in the lecture of 29-01-2018.
(c) Write down the invader dynamics for the strategy \( y \) and corresponding population density \( m \), and give an expression for the invasion fitness.

(d) Use the principle of selective neutrality of residents to show that coexistence of different resident strategies is not possible except under very special conditions. What are these special conditions?

(e) Show that the invasion fitness in a monomorphic resident population with strategy \( x \) and corresponding population density \( n \) is positive if and only if

\[
\frac{\alpha(y)}{r(y)} < \frac{\langle 1 - n(t) \rangle}{\langle \frac{p(t)}{1 + \beta(x)n(t)} \rangle}
\]

where \( \langle \ldots \rangle \) is the longterm time average of its argument.

(f) Use the principle of selective neutrality of residents to show that \( \alpha(x)/r(x) \) of successive residents decreases with every invasion-substitution event.