The existence of a pair of recursively enumerable, recursively *inseparable* sets

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Let A and B be a pair of recursively enumerable, disjoint sets of natural numbers. A and B are said to be *recursively inseparable* if for no recursive set C do we have $A \subseteq C$ and $C \cap B = \emptyset$.

Claim Let $A = \{\ulcorner \phi \urcorner \,|\, PA \vdash \phi\}$, $B = \{\ulcorner \psi \urcorner \,|\, PA \vdash \neg\psi\}$. Then A and B are recursively inseparable.

Proof Suppose otherwise. Let C be a recursive set containing A and disjoint from B. Let θ represent C in PA, i.e.

$$n \in C \rightarrow PA \vdash \theta(\overline{n})$$

$$n \notin C \rightarrow PA \vdash \neg\theta(\overline{n}).$$

By the Fixed Point Lemma, let ψ be a fixed point of $\neg\theta$, i.e.

$$PA \vdash \neg\theta(\overline{\ulcorner \psi \urcorner}) \leftrightarrow \psi.$$

Case 1. $\ulcorner \psi \urcorner \in C$. Then $PA \vdash \theta(\overline{\ulcorner \psi \urcorner})$. But then since ψ is a fixed point of $\neg\theta$, $PA \vdash \neg\psi$. But then $\ulcorner \psi \urcorner \in B$, a contradiction since $C \cap B = \emptyset$.

Case 2. $\ulcorner \psi \urcorner \notin C$. Then $PA \vdash \neg\theta(\overline{\ulcorner \psi \urcorner})$, i.e. $PA \vdash \psi$ (as before, because ψ is a fixed point of $\neg\theta$). But then $\ulcorner \psi \urcorner \in A$, and hence $\ulcorner \psi \urcorner \in C$, a contradiction.