Computability theory

Exercise 4

Let $x \in \mathbb{N}$ and $x > 0$. Then x has unique representations as

$$x = 2^{b_1} + 2^{b_2} + \cdots 2^{b_l}, \text{ where } 0 \leq b_1 < b_2 < \cdots < b_l \text{ and } l \geq 1 \tag{1}$$

and

$$x = 2^{a_1} + 2^{a_1+a_2+1} + \cdots + 2^{a_1+\cdots+a_l+l-1}. \tag{2}$$

Define $\alpha(i, x)$ as follows:

$$\alpha(i,x) = \begin{cases} 1 & \text{if } 2^i \text{ appears as a summand in (1)} \\ 0 & \text{if } 2^i \text{ does not appear as a summand in (1)} \end{cases}$$

Furthermore, define the functions l, b, and a as follows:

$$l(x) = \begin{cases} l \text{ as in (1)} & \text{if } x > 0, \\ 0 & \text{otherwise} \end{cases}$$

$$b(i,x) = \begin{cases} b_i \text{ as in (1)} & \text{if } x > 0 \text{ and } 1 \leq i \leq l \\ 0 & \text{otherwise} \end{cases}$$

$$a(i,x) = \begin{cases} a_i \text{ as in (2)} & \text{if } x > 0 \text{ and } 1 \leq i \leq l \\ 0 & \text{otherwise} \end{cases}$$

1. Show that the functions α and l above are computable.

2. Show that the functions b and a above are computable.

3. Let $f(x)$ be a total injective computable function. Show that f^{-1} is also computable.

4. Let $p(x)$ be a polynomial with integer coefficients. Show that the function $f(a)$ defined below is computable:

$$f(a) = \begin{cases} \text{the least non-negative integral root of } p(x) - a, & \text{if such a root exists} \\ \text{undefined,} & \text{otherwise} \end{cases}$$

5. Devise a Turing machine that computes the function $2x$.