Computability theory
Exercise 5

1. Recall that every $x \in \mathbb{N}$ has a unique representation in the form

$$x = 2^{b_0} + 2^{b_1} + \cdots + 2^{b_{l-1}},$$

where $0 \leq b_0 < b_1 < \cdots < b_{l-1}$. This representation can be encoded by a binary string $bin(x) \in \{0,1\}^*$ in the usual way. On the other hand, every $x \in \mathbb{N}$ has a unique dyadic (2-adic) representation in the form $k_0 \cdots k_{l-1} \in \{1,2\}^*$ such that

$$x = k_0 2^0 + k_1 2^1 + \cdots + k_{l-1} 2^{l-1},$$

where $k_i \in \{1,2\}$. Compare these two representations of natural numbers with each other.

2. Let f and g be computable functions. Argue using Church's thesis that the function h defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathrm{Dom}(f) \cap \mathrm{Dom}(g) \\ \text{undefined,} & \text{otherwise} \end{cases}$$

is URM-computable.

3. Let $f(x)$ be a total unary computable function. Show using Church's thesis that the function h defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathrm{Ran}(f) \\ \text{undefined,} & \text{otherwise} \end{cases}$$

is URM-computable.

4. Let $\zeta \colon \mathbb{N}^+ \times \mathbb{N}^+ \times \mathbb{N}^+ \to \mathbb{N}$ be the bijection defined by

$$\zeta(m,n,q) = \pi(\pi(m-1,n-1),q-1).$$

Show that the function ζ^{-1} defined by

$$\zeta(x) = (\pi_1(\pi_1(x)) + 1, \pi_2(\pi_1(x)) + 1, \pi_2(x) + 1)$$

is the inverse of ζ. Here π denotes the computable pairing function of natural numbers (also $\pi^{-1}(x) = (\pi_1(x), \pi_2(x))$) discussed in the lectures (see also Exercise 3 and p. 73 of the textbook).

5. Find

a) $\beta(J(3, 4, 2))$

b) $\beta^{-1}(503)$

c) P_{100}