Computability theory

Exercise 8

1. Let $\vec{x} = (x_1, \ldots, x_n)$ and consider the $(n+3)$-ary function f defined by

$$\begin{aligned} f(e_1, e_2, \vec{x}, 0) &\simeq \phi_{e_1}^{(n)}(\vec{x}), \\ f(e_1, e_2, \vec{x}, y+1) &\simeq \phi_{e_2}^{(n+2)}(\vec{x}, y, f(e_1, e_2, \vec{x}, y)). \end{aligned}$$

Show that f is computable and show using it that there exists a total computable function $r(e_1, e_2)$ such that

$$\phi_{r(e_1,e_2)}^{(n+1)} \simeq Rec(\phi_{e_1}^{(n)}, \phi_{e_2}^{(n+2)}),$$

where $Rec(f, g)$ denotes the function obtained by recursion from f and g.

2. Show that there is a total computable function $k(e)$ such that for every x $E_{k(x)} = W_x$.

3. Show that there is a total computable function $k(e)$ such that, for any e, if ϕ_e is the characteristic function of a decidable predicate $M(x)$ then $\phi_{k(x)}$ is the characteristic function of the complement of $M(x)$ (i.e., the predicate 'not $M(x)$').

4. Suppose that $f(x)$ is computable. Show that there is a total computable function $k(x)$ such that for all x, $W_{k(x)} = f^{-1}(W_x)$.

5. Show that the predicate '$x \in E_x$' is undecidable.

6. Show that the predicate '$W_x = W_y$' is undecidable.