Applications of matrix computations, Spring 2018 Exercise 2

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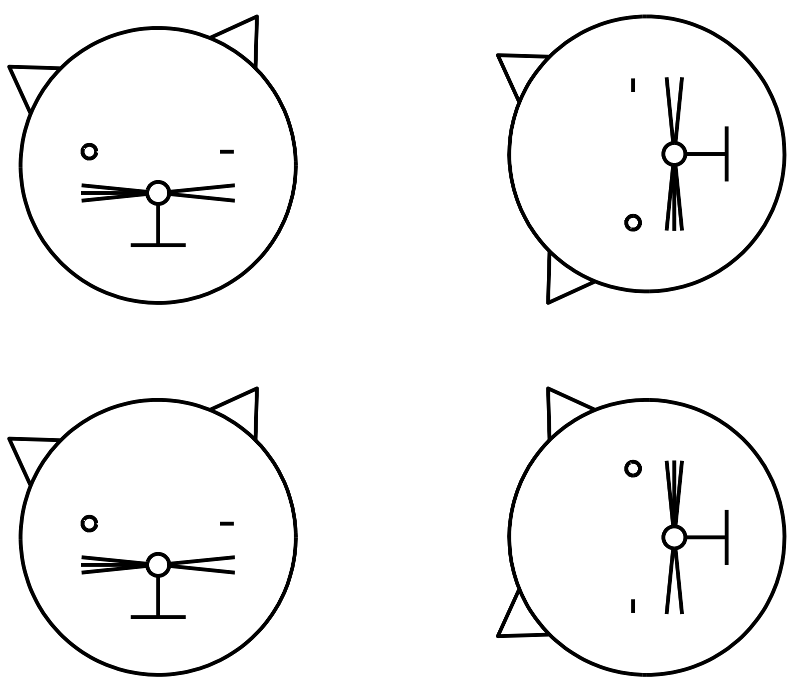
Deadline: Monday 09.04. at 10.00. *Sending your solutions late is not allowed!*

The exercises are held in the room C321 of Exactum on Wed 28.3. 14.15–16.00 (group 1), Thu 5.4. 12.15–14.00 (group 2), Fri 6.4. 10.15–12.00 (group 3), and Fri 6.4. 14.15–16.00 (group 4). (Notice the Easter break!) You gain credits from this course only by completing the weekly exercises. Participating in the exercise group is not obligatory for completing the course, but it is highly recommended.

Return your solutions as a single PDF file containing your Matlab codes (as text), results, images, comments, and explanations to the email address application.matrixcomputation@gmail.com before the deadline. Please include your name, student number, and your exercise group number in the beginning of the PDF file.

1. Download the routine `kissa.m` from the course website. Find the two 2×2 matrices producing the following outputs with the `kissa.m`:



2. Define a matrix A by

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}.$$

   (a) Determine the eigenvalues and eigenvectors of A using the Matlab command `eig`.

   (b) Using the information from (a), write A in the form $A = PDP^{-1}$ where D is diagonal.

   (c) Demonstrate the action of the matrices P and D using the routine `kissa.m`. Explain what you see.

3. *Canonical form of a* 2×2 *matrix with complex eigenvalues.* Define

$$A = \begin{bmatrix} 5 & -6 \\ 3 & 1 \end{bmatrix}.$$

(a) Use the command `[V,D]=eig(A)` to find the eigenvalues $\lambda = a + ib$ and $\overline{\lambda} = a - ib$ and a complex-valued matrix V.

(b) Construct in Matlab the matrix

$$R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

(c) Construct in Matlab the matrix P whose (1) first column is the real part of the first column of V and (2) the second column is the imaginary part of the second column of V. Check numerically that $A = PRP^{-1}$.

4. (a) Use the command `meshgrid` to create these two matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}.$$

Hint: type `help meshgrid` into Matlab.

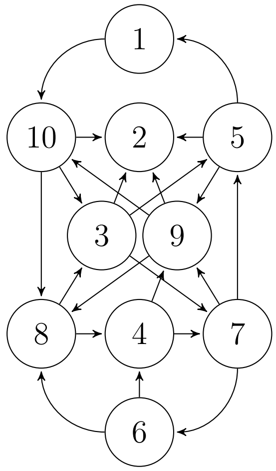
(b) Consider the following function of two real variables: $e^{-(x^2+y^2)/(2\sigma^2)}$ in the square defined by the conditions $-1 \le x \le 1$ and $-1 \le y \le 1$. Create a plot with the following commands:

```
sigma = .2;
t     = linspace(-1,1,128);
[x,y] = meshgrid(t);
fun = exp(-(x.^2+y.^2)/(2*sigma^2));
figure(1)
clf
mesh(x,y,fun)
zlim([0 1])
```

Experiment a little with the parameter $\sigma > 0$. Which value of σ gives, in your opinion, the most beautiful picture of the Gaussian bell curve?

(c) Plot the most beautiful bell curve from (b) in the square defined by the conditions $-1 \le x \le 1$ and $4 \le y \le 6$. Hint: you will need two different `linspace` commands.

5. Consider the following ten-page internet, where arrows show links:



Use the PageRank algorithm as follows to find out the most important page.

(a) Construct the adjacency matrix $A = [a_{ij}]$, where i is row index and j is column index and

$$a_{ij} = \begin{cases} 1 & \text{if page } j \text{ has a link to page } i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Use the power method https://en.wikipedia.org/wiki/Power_iteration to find the dominant eigenvalue of A and the corresponding eigenvector. Compute the eigenvalues and eigenvectors using the command `eig` and check that the result of the power method is approximately right.

(c) Normalize the eigenvector calculated in (b). Which web page is the most important?

6. Consider the Markov chain given by the teleoperator market share application discussed in the lecture. The explanation of the application is given on the course web page in the file *TeleOperatorComputation.pdf*.

We model the market situation by the horizontal vector $x = [x_1, x_2, x_3, x_4] \in \mathbb{R}^4$. Every component of x satisfies $0 \leq x_j \leq 1$, and additionally

$$x_1 + x_2 + x_3 + x_4 = 1.$$

The interpretation is that x_j is the market share of company j at a time.

Given an initial state $x^{(0)}$, we get the market shares of the next day by

$$x^{(1)} = x^{(0)} P,$$

where $P = [p_{ij}]$ is the *transition matrix* with elements defined by

$$p_{ij} = \text{probability of a customer to change from company } i \text{ to company } j.$$

Now it is easy to see that on day n the market shares are given by $x^{(n)} = x^{(0)} P^n$.

Take

$$P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{10} & 0 & 0 & \frac{9}{10} \end{bmatrix}$$

Note that the matrix P above differs from the one in the file *TeleOperatorComputation.pdf*. This is done on purpose.

(a) Take $x^{(0)} = [\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$ and compute $x^{(n)}$ for several values of n. How big needs n to be for the ten first digits of $x^{(n-1)}$ and $x^{(n)}$ to agree? Hint: use the command `format long` before the computation.

(b) Choose a couple of other initial states $x^{(0)}$ and compute again $x^{(n)}$ for several values of n. How big needs n to be for the ten first digits of $x^{(n-1)}$ and $x^{(n)}$ to agree?

(c) After the experiments (a) and (b) you should be convinced that there is a unique equilibrium state $x^{(\infty)} = \lim_{n\to\infty} x^{(n)}$. Find out how to write $x^{(\infty)}$ as an eigenvector of a matrix, and compute $x^{(\infty)}$ using Matlab's `eig` command.