Computability theory
Exercise 9

1. Let $n \geq 1$. Show that the predicate '$E_x^{(n)} \neq \emptyset$' is partially decidable.

2. Let $M(\vec{x}, y)$ be a partially decidable predicate. Show that the predicate $\forall y M(\vec{x}, y)$ need not be partially decidable.

3. Let $M(x)$ be a predicate such that there exists a total computable function $k(x)$ such that for all x

$$x \in W_x \Leftrightarrow M(k(x)) \text{ does not hold.}$$

Show that $M(x)$ is not partially decidable.

4. Show that the predicate 'ϕ_x is not total' is not partially decidable (Hint: use Exercise 3).

5. By considering the function

$$f(x, y) = \begin{cases} 1 & \text{if } P_x(x) \text{ does not halt in } y \text{ or fewer steps} \\ \text{undefined} & \text{otherwise} \end{cases}$$

show that the predicate 'ϕ_x is total' is not partially decidable. (Hint: use Exercise 3 and the s-m-n Theorem)

6. Let $A, B \subseteq \mathbb{N}$ such that $A \leq_m B$ and B is recursively enumerable. Show that A is also recursively enumerable.