Applications of matrix computations, Spring 2018 Exercise 5

Lecturer: Samuli Siltanen.
Teaching assistants: Vesa Kaarnioja and Jesse Railo.

Deadline: Monday 30.04. at 10.00. *Sending your solutions late is not allowed!*

The exercises are held in the room C321 of Exactum on Wed 14.15–16.00 (group 1), Thu 12.15–14.00 (group 2) and Fri 14.15–16.00 (group 4), and in the computer lab C128 of Exactum on Fri 10.15–12.00 (group 3). You gain credits from this course only by completing the weekly exercises. Participating in the exercise group is not obligatory for completing the course, but it is highly recommended.

Return your solutions as a single PDF file containing your Matlab codes (as text), results, images, comments, and explanations at the Moodle page of the course before the deadline (further instructions are given at the exercise groups if needed). Please include your name, student number, and your exercise group number in the beginning of the PDF file.

1. **How to calculate Fourier series using the FFT?** Consider a vector $\mathbf{f} = [\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n]^T \in \mathbb{R}^n$. The discrete Fourier transform of $\mathbf{f}$, denoted here by $\widehat{\mathbf{f}}$, is the following n-dimensional *complex-valued* vector:

$$\widehat{\mathbf{f}}_k := \sum_{j=1}^{n} \mathbf{f}_j \exp(-2\pi i(k-1)(j-1)/n), \qquad 1 \leq k \leq n. \tag{1}$$

   (a) Compute the one-dimensional FFT of some nonzero vector of length 128 with non-negative elements. Plot the absolute value of the transformed signal both with and without using `fftshift`. Where is the zero-frequency component located in those two cases? Why?

   Hint: you can use “`>>help fft`” in Matlab.

   (b) In the lectures we discussed 2π-periodic functions $f(\theta)$ defined for $\theta \in [0, 2\pi]$ and satisfying $f(0) = f(2\pi)$. Furthermore, we defined Fourier series coefficients

$$\widehat{f}(\nu) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(\theta) e^{-i\nu\theta} d\theta \tag{2}$$

   for all $\nu \in \mathbb{Z}$. What is the onnection between the DFT coefficient $\widehat{\mathbf{f}}_k$ given by (2) and the number $\widehat{f}(\nu)$ defined in (1)?

   Hint: discretize the interval $[0, 2\pi]$ as $\theta_j = (j-1)\Delta\theta$ with $\Delta\theta := 2\pi/n$ and $j = 1, \ldots, n$. Set $\mathbf{f}_j := f(\theta_j)$ and use the midpoint rule to discretize the integral in (2).

2. **Theory for the FFT.** In this problem you need to study problems given in the lecture notes *Fourier series, Haar wavelets and Fast Fourier transform* available on the course webpage. Let $A_N = \{0, \dots, N-1\}$, $e(t) := e^{2\pi i t}$, and $\mathcal{F}$ be the DFT operator. (You can include a picture containing your calculations to your final pdf file.)

   (a) Show that $\mathcal{F}$ is a linear mapping. (Exercise 4.1)

   (b) Let us complete the proof for the inverse DFT formula. Verify that

   $$\sum_{\xi \in A_N} e\left(\frac{x\xi}{N}\right) = \begin{cases} 0 & \text{if } x \neq 0 \\ N & \text{if } x = 0. \end{cases}$$

   Basic formulas for the geometric series might be useful. (Exercise 4.3)

   (c) Find $C > 0$ such that $\|\mathcal{F}f\| = C\,\|f\|$ holds for every $f \in \mathbb{C}^N$. (Hint: Look at the page 9 and consider the DFT matrix. You can use the general fact for orthogonal matrices (note that this fact was explained in the notes on *Least squares solutions*): if $A^{-1} = A^*$, then $\|Af\| = \|f\|$, where A^* denotes the conjugate transpose of A, called also an adjoint.)

   (d) Let $N \in \mathbb{Z}_+$ be fixed and assume that all the values of $e^{-2ij\pi k/N}$ are stored in a database for $j, k = 0, \dots, N-1$. Let $f \in \mathbb{C}^N$ (i.e. a function $A_N \to \mathbb{C}$). How many elementary operations (products and sums) is at most needed to calculate $\mathcal{F}f$ directly from the definition given at the end of the page 7? Explain based on Theorem 5.2 why the FFT algorithm is faster than the direct calculation. (Exercise 5.4)

   (Solving 2 out of the following: (a)+(b), (c) or (d) gives 1 point from this Problem.)

3. **Reducing Gibbs phenomenon** using a filtering trick related to the infamous Fejér kernel. The file *FourierSeries1.m* from the lecture on 18.4.2018 may be useful.

   (a) Use the complex Fourier series with a sequence of ever larger N, such as $N = 100, 200, 300, ...$, to approximate a function with jump discontinuities. Notice the rapid oscillations near the jumps (Gibbs phenomenon).

   (b) Apply the following filter to the coefficients: $c_n' = c_n(1 - |n|/N)$. What do you observe when you approximate $f(x)$ with

   $$\sum_{n=-N}^{N} c_n'\,\varphi_n(x)?$$

4. **Numerical study of computing times.** In the following create Matlab .m files for the functions. (You need to play with `for` loops a lot in this exercise.)

   (a) Create a Matlab function that takes $N \in \mathbb{Z}_+$ as an input and gives the DFT matrix `A` as an output. Choose a nonzero vector $x \in \mathbb{R}^{128}$ (e.g. from (a) of Problem 1) and test that `Ax` $\approx$ `fft(x)`.

   (b) Create a Matlab function that calculates the DFT directly from the definition using a `for` loop (inputs: the DFT matrix and a vector x, output: $\mathcal{F}x$.).

   (c) Store in the memory the DFT matrices for the values $N = 2^\alpha$ and vectors `x = ones(1,`2^α`)` where $\alpha = 1, \ldots, 10$. Study how much it takes computing time to calculate $\mathcal{F}x$ using the DFT matrices, direct DFT and FFT for various values of N. What do you observe? (Hint: look at `help tic`, `help toc`. You can visualise the computing times as functions of α or N.)

5. **Band-pass filtering of speech.** Download from the course website the Matlab file *Fourierseries_realsoundFFT_test.m* and use it as your starting point.

   (a) Record a short sentence using a laptop or a smartphone. Read the sound file into Matlab, truncate it to a power-of-two length 2^m and play it with the command `sound`.

   (b) Band-pass filter the signal using the index vector
   `index = (abs(nvec)<F1)|(abs(nvec)>F2);`
   Experiment with choices of the cutoff frequencies `F1` and `F2`.

   (c) Adjust the cutoff frequencies `F1` and `F2` so that they are as close to each other as possible but you can still understand the sentence. How small can you make the compression ratio $(F_2 - F_1)/2^m$?

6. **Sound signal analysis using FFT.** Download the file *phone1234567890.m4a* from the course website. It contains the sounds of dialing $1, 2, 3, \ldots, 9, 0$ on a cell phone. The Matlab routine `PhoneSoundAnalysis.m` will open the file and extract the sounds of each number.

   (a) Modify the file `PhoneSoundAnalysis.m` so that it plots the FFT of each sound. Plot them on top of each other using
   `subplot(10,1,1)`, `subplot(10,1,2)`, $\ldots$, `subplot(10,1,10)`
   for easy comparison of location of frequency components. This way you should be able to recognize every number by its unique frequency content.

   (b) Download the file *MysteryNumber.m4a*. It is the sound of dialing a certain phone number with eight digits. Using the results of (a), find out the mysterious phone number.