Computability theory
Exercise 12

1. Show that $\{\emptyset\}$ and $\{\mathbb{N}\}$ are m-degrees.

2. Let $\chi\colon \mathbb{N} \to \mathbb{N}$ and $K^\chi = \{x \mid x \in W_x^\chi\}$. Show that K^χ is χ-r.e. but not χ-recursive.

3. Let A and B be such that there are total injective computable functions f and g such that $f\colon A \leq_m B$ and $g\colon B \leq_m A$. Show that A and B are *recursively isomorphic*, that is, there exists a computable bijection $I\colon \mathbb{N} \to \mathbb{N}$ such that $I(A) = B$. (Hint: Define I inductively and apply Church's thesis)

4. Let $K^1 := K$ and $K^{i+1} := \{x \mid x \in W_x^{K^i}\}$. Show that for all n there exists a decidable predicate $R(x, y_1, \ldots, y_n)$ such that

$$x \in K^n \Leftrightarrow \exists x_1 \forall x_2 \exists x_3 \cdots Q x_n R(x, y_1, \ldots, y_n),$$

where $Q = \exists$ if n is odd, and $Q = \forall$ otherwise.