Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 1 / September 3, 2018
due by 17.9.2018 (before the exercise class)
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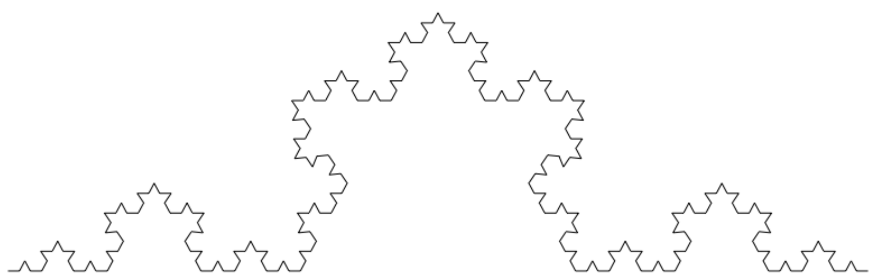
1. For $\alpha, \beta > 0$, let $f : \mathbb{R} \to \mathbb{R}$ that $f(x) = x^\alpha$ if $x > 0$ and $f(x) = -|x|^\beta$ for $x < 0$. Show that f is quasisymmetric if and only if $\alpha = \beta$.

   (Note that f and f^{-1} are Hölder continuous mappings for any $\alpha, \beta > 0$.)

2. (a) Show that the standard von Koch snowflake K (as described on page 5-6 in the lecture notes) may be represented as the image of the line segment $[0, 1]$ under an η-quasisymmetric map $f : [0, 1] \to K$ with the form of $\eta(t) = Ct^\alpha$ for some $\alpha < 1$.

   (b) Can you tell what α is ?

   (c) Show that the map f above is not differentiable at any point $x \in [0, 1]$.



3. Prove that if $f : \mathbb{D} \to \mathbb{C}$ is a conformal mapping with $f(\mathbb{D}) = \mathbb{D}$, then f is a Möbius transformation. That is, f is of the form

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z},$$

   for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.

4. Let

$$f(z) = z + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \dots$$

   be a conformal map of the exterior unit disk $\{z : |z| > 1\}$ to a domain $\Omega = \mathbb{C} \setminus K$. The aim of this problem is to give an alternative proof of the area theorem (Theorem 2.4). That is, find a formula for the area of K in terms of the coefficients b_k by following the steps below.

(a) Find an asymptotic formula (as $\rho \to \infty$) for the area $A(\rho)$ of the compact set enclosed by the curve $f(S_\rho)$, where $S_\rho = \{z : |z| = \rho\}$. *Hint.* Use the geometry of the set $\{z + \frac{b_1}{z} : |z| = \rho\}$.

(b) Compute

$$B_r(\rho) = \text{Area}(f(r < |z| < \rho)) = \int_{r<|z|<\rho} |f'(z)|^2,$$

by using the Laurent series expansion of f.

(c) Observe that

$$\text{Area}(\mathbb{C} \setminus f(|z| > r)) = A(\rho) - B_r(\rho)$$

for any $\rho > r$. Analyze the limit $\rho \to \infty$ and finally find the area of K by taking the limit $r \to 1$.