Introduction to Quantum Computation
Master's Programme in Mathematics and Statistics
Fall 2018
Exercise set 1

Linear algebra recap. If you have forgotten the concepts and techniques, go back to your favourite course or reference material, or look them up in the wild world of the web.

Exercise 1. Consider the complex two-dimensional vector space $\mathbb{C}^2$ and let the vectors $|0\rangle$ and $|1\rangle$ form an orthonormal basis for the space (think e_0 and e_1 if the notation is confusing at this point).

(a) Let A be a linear operator from $\mathbb{C}^2$ to $\mathbb{C}^2$ such that $A|0\rangle = |1\rangle$ and $A|1\rangle = |0\rangle$. Give a matrix representation for A with respect to input and output basis $\{|0\rangle, |1\rangle\}$.
(b) What linear operator does the matrix

$$B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

correspond to?
(c) What is the length of $B|0\rangle$?
(d) What is the matrix representation for the identity operator?

Exercise 2. Suppose U, V and W are finite vector spaces and $A : U \to V$ and $B : V \to W$ are linear operators. Show that the matrix representation for the linear transformation BA is the matrix product of the matrix representations for B and A, with respect to appropriate bases.

Generalising inner product spaces to vector spaces over $\mathbb{C}$: If V is a vector space over $\mathbb{C}$, an inner product is a function $(\cdot,\cdot) : V \times V \to \mathbb{C}$ such that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $c, d \in \mathbb{C}$:

(a) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})^*$, *where c^* is the* complex conjugate *of c,*
(b) $(\mathbf{u}, c\mathbf{v} + d\mathbf{w})) = c(\mathbf{u}, \mathbf{v}) + d(\mathbf{u}, \mathbf{w})$, *i.e. the inner product is linear in the second argument,*
(c) $(\mathbf{v}, \mathbf{v}) \geq 0$ *with equality if and only if* $\mathbf{v} = 0$.

Exercise 3.

(a) Verify that the inner product defined on $\mathbb{C}^n$ by

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) = \sum_i a_i^* b_i$$

is, indeed, an inner product.
(b) Show that any inner product is conjugate-linear in the first argument, i.e.,

$$(c\mathbf{u} + d\mathbf{v}, \mathbf{w}) = c^*(\mathbf{u}, \mathbf{w}) + d^*(\mathbf{v}, \mathbf{w}).$$

Exercise 4. The vectors $(1,0)$ and $(1,-1)$ in $\mathbb{C}^2$ span the whole two-dimensional space, but are not orthogonal to each other. Use the Gram-Schmidt procedure on these two vectors to produce two different orthonormal bases for $\mathbb{C}^2$.

Exercise 5. Find the eigenvalues and eigenvectors of the operators corresponding to the matrices

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Exercise 6. We look at the vector space $\mathbb{C}^4$. What is the orthogonal projection of

(a) the vector $\mathbf{u} = (1,0,-i,2)$ onto the vector space spanned by the vectors
$$(1,0,0,0),\ (1,2,0,0) \text{ and } (i,2,i,0)?$$

(b) the vector $\mathbf{v} = (1,0,i,0)$ onto the vector space spanned by the vectors
$$(0,3,5,2),\ (3,0,-1,2) \text{ and } (5,1,0,4)?$$