Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 2 / September 5, 2018
due by 24.9.2018 (before the exercise class)
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1. Show that any quasisymmetric map $f : \mathbb{C} \to \mathbb{C}$ is surjective.

2. Let $f(z) = |z|^\alpha$, $z \in \mathbb{C}$ and $\alpha > -1$. If $g(z) = (\alpha/2)\overline{z}|z|^{\alpha-2}$, show that

$$\int_{\mathbb{C}} f(z)\partial\phi(z)dm(z) = -\int_{\mathbb{C}} g(z)\phi(z)dm(z)$$

for every $\phi \in C_0^\infty(\mathbb{C})$. Conclude that the weak derivative $\partial f = g$.

Hint. Use Green's formula

$$\int_{\partial D} hd\overline{z} = -\frac{1}{2i}\int_D \partial h dm, \qquad h \in C^\infty(\overline{D}),$$

with $D = D_\epsilon = \{z : \epsilon < |z| < R\}$ and take $\epsilon \to 0$.

A word on notation. As is standard, we use the abbreviation $\partial f := \frac{\partial f}{\partial z} = \frac{1}{2}(\partial_x f - i\partial_y f)$.

3. Given a function $f : \mathbb{R} \to \mathbb{R}$, let

$$L_f^\epsilon(z) = \sup_{|h|<\epsilon} \frac{|f(z+h) - f(z)|}{h}.$$

   (a) Show that $L_f^\epsilon \in L^1_{\text{loc}}$ implies that f is absolutely continuous.
   (b) State the analogues of Lemma 3.8 and Corollary 3.9 from the lecture notes in the 1-dimensional case. In particular, decide if the arguments imply that "all quasisymmetric maps of the real line are absolutely continuous."

4. The *Lebesgue density theorem* states that for a set $E \subset \mathbb{R}$, and almost any point $x \in E$,

$$\frac{m(E \cap B(x,r))}{m(B(x,r))} \to 1, \qquad \text{as } r \to 0.$$

This is a special case of the *Lebesgue differentiation theorem* which says that for a function $f \in L^1$,

$$\lim_{r \to 0} \frac{\int_{B(x,r)} f(y) dm(y)}{m(B(x,r))} \to f(x), \quad \text{for a.e. } x.$$

The aim of this exercise is to prove the above theorems.

(a) Show that the density theorem follows from the differentiation theorem.

(b) Observe that the Lebesgue differentiation theorem is valid for continuous functions.

Consider the maximal function

$$Mh(x) = \sup_{r>0} \frac{\int_{B(x,r)} h(y) dm(y)}{B(x,r)}.$$

A useful fact is the estimate

$$m(\{Mh > t\}) < C/t \cdot \|h\|_{L^1}.$$

(You are not asked to prove this estimate here.)

(c) Suppose $\epsilon > 0$. Using an approximation of an L^1 function by continuous functions $g_n \to f$ in L^1, and $h_n = f - g_n$, show that the set of points x where

$$\limsup_{r \to 0} \frac{\int_{B(x,r)} f(y) dm(y)}{m(B(x,r))} > f(x) + \epsilon,$$

has measure 0.

(d) Similarly, show that the set of points x where

$$\liminf_{r \to 0} \frac{\int_{B(x,r)} f(y) dm(y)}{m(B(x,r))} < f(x) - \epsilon,$$

has measure 0.