Finite model theory
Problems 1
Tuesday 11.9.2018

1. Give examples of the following types of binary relations R on a set X:

1. R is reflexive, symmetric, but not transitive.
2. R is a partial-order but not a linear-order.

2. Let X be a set of cardinality n, and $k \in \mathbb{N}$. Determine the number of k-ary relations over X. How many of those are symmetric?
3. Let τ be a finite vocabulary. Show that the number of non-isomorphic τ-models of cardinality n is bounded by $2^{p(n)}$, where $p(x)$ is a polynomial function.
4. Let f be a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$, and g a homomorphism from $\mathfrak{B}$ to $\mathfrak{C}$. Show that $g \circ f$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{C}$.
5. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\{f\}$-models, where f is an n-ary function symbol. Let $R_f^{\mathfrak{A}}$ and $R_f^{\mathfrak{B}}$ denote the graphs of the functions $f^{\mathfrak{A}}$ and $f^{\mathfrak{B}}$, that is

$$R_f^{\mathfrak{A}} = \{(\overline{a}, f(\overline{a})) : \vec{a} \in \mathrm{Dom}(\mathfrak{A})^n\}$$

and $R_f^{\mathfrak{B}}$ is defined analogously. Let $\mathfrak{A}^*$ and $\mathfrak{B}^*$ be obtained from $\mathfrak{A}$ and $\mathfrak{B}$ by replacing the functions $f^{\mathfrak{A}}$ and $f^{\mathfrak{B}}$ by their graphs, i.e, $\mathfrak{A}^*$ is the $\{R\}$-model (where R has arity $n+1$) such that $\mathrm{Dom}(\mathfrak{A}^*) = \mathrm{Dom}(\mathfrak{A})$ and $R^{\mathfrak{A}^*} = R_f^{\mathfrak{A}}$. Let $h\colon \mathrm{Dom}(\mathfrak{A}) \to \mathrm{Dom}(\mathfrak{B})$ be a function. Show that h is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if and only if h is a homomorphism from $\mathfrak{A}^*$ to $\mathfrak{B}^*$.
6. Let $\mathbb{G} = (V, E)$ be a graph of cardinality at least 6. Show that there exists $a, b, c \in V$ such that either $\{(a,b),(b,c),(c,a)\} \subseteq V$ or $\{(a,b),(b,c),(c,a)\} \subseteq V^c$, where $V^c = V^2 - E$.