Department of Mathematics and Statistics
Quasiconformal mappings
Exercise Set 3 / September 17, 2018
due by 1.10.2018 (before the exercise class)
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1. Let $f\colon \Omega \to \mathbb{C}$ be differentiable at a point $z \in \Omega$.

   (a) Show that with the complex derivatives $\partial f(z)$ and $\bar{\partial} f(z)$, the derivative takes the form
   $$Df(z)h = \partial f(z)h + \bar{\partial} f(z)\bar{h}.$$

   (b) Show that the derivatives of $f$ and $\bar{f}$ are related by $\overline{\partial f(z)} = \bar{\partial}\bar{f}(z)$. Show also, that the Jacobian determinant $J(z,f) \equiv \det Df(z) = |\partial f(z)|^2 - |\bar{\partial} f(z)|^2$.

   (c) Show that if, in addition, $g$ is differentiable at $f(z)$, then the chain rule obtains the form
   $$\bar{\partial}(g \circ f)(z) = (\partial g)(fz)\bar{\partial} f(z) + (\bar{\partial} g)(fz)\overline{\partial f(z)},$$
   $$\partial(g \circ f) = (\partial g)(fz)\partial f(z) + (\bar{\partial} g)(fz)\overline{\bar{\partial} f(z)}.$$

2. Suppose $u \in C^2(\overline{\Omega})$ where $\Omega = \{z : r < |z| < 1\}$, with boundary values $u(re^{i\theta}) = 0$, $u(e^{i\theta}) = 1$, $0 \le \theta \le 2\pi$. Find the optimal lower bound for the energy
   $$\mathscr{E}(u) = \int_\Omega |\nabla u|^2 dxdy.$$

3. According to Theorem 3.11 in the notes, if $f : \Omega \to \Omega'$ is a $\eta$-quasisymmetric map between bounded domains in $\mathbb{R}^2$, then
   $$L_f(z_0) := \limsup_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}$$
   is in $L^2_{\text{loc}}$, which is slightly better than the weak-$L^2_{\text{loc}}$ estimate for
   $$L^\epsilon_f(z_0) := \sup_{|z-z_0| \le \epsilon} \frac{|f(z) - f(z_0)|}{|z - z_0|}.$$

   (a) Show that in the one dimensional case, the analogous function $L_f(x) \in L^1$.

(b) It would appear that $\int_a^b L_f^\epsilon \to \int_a^b L_f$ converge as $\epsilon \to 0$ (since the functions $L_f^\epsilon$ are decreasing in $\epsilon$), suggesting that $f$ is absolutely continuous. However, it is known that not every quasisymmetric map $f\colon \mathbb{R} \to \mathbb{R}$ is absolutely continuous. Identify the point where the arguments in (a-b) break down.

4. Let $u : \Omega' \to \mathbb{R}$ be a Lipschitz function, that is there exists a constant $L < \infty$ such that $|u(z) - u(w)| \leq L|z - w|$ for all $z, w \in \Omega'$. Let $1 \leq p < \infty$ and $f \in W^{1,p}_{\text{loc}}(\Omega)$ with $f(\Omega) \subseteq \Omega'$. Show that $u \circ f \in W^{1,p}_{\text{loc}}(\Omega)$.

   *Hint:* use the ACL definition of Sobolev spaces.