Introduction to Quantum Computation
Master's Programme in Mathematics and Statistics
Fall 2018
Exercise set 3

Exercise 1. A qubit is measured with respect to the computational basis. After this, a Hadamard transformation (given by the matrix $\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$) is applied to it. Then it is measured again with respect to the computational basis. What is the probability of observing a 0 in the second measurement?

Exercise 2.

(a) $(A \otimes B)(C \otimes D) = AC \otimes BD$,
(b) $(A + B) \otimes C = A \otimes C + B \otimes C$ and $A \otimes (C + D) = A \otimes C + A \otimes D$,
(c) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$,
(d) if A and B are unitary, then so is $A \otimes B$.

Exercise 3. Let V and W be complex inner product spaces with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_W$, respectively. Show that the formula $(\sum_i a_i v_i \otimes w_i, \sum_j b_j v'_j \otimes w'_j) = \sum_{i,j} a_i^* b_j (v_i, v'_j)_V (w_i, w'_j)_W$ defines an inner product on $V \otimes W$.

Exercise 4. Show that if we measure the first qubit of the Bell state $|\beta_{00}\rangle$ and observe a 0, then after this the probability of observing a 0 when measuring the second qubit is 1.

Exercise 5. There are four Bell states

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

Show that the state β_{xy} is obtained from the two-qubit basic state $|xy\rangle$ by first applying a Hadamard operator to the first qubit, and then a controlled `NOT` operator to the qubits. The controlled `NOT` does nothing if the first qubit is 0, and flips the second qubit if the first is 1.

Exercise 6. A two-qubit system is in the state

$$|\psi\rangle = \frac{1}{\sqrt{31}}(|00\rangle + i\,|01\rangle + 2i\,|10\rangle + (3+4i)\,|11\rangle).$$

The first qubit is measured in the computational basis and the outcome is 1. What is the state of the system after the measurement? What is the probability of observing a 1 if next the second qubit is measured?