THIRD EXERCISES FOR GMT

Exercise 1 (1 point)**.** Let $\{\varphi_i\}_{i\in\mathbb{N}}\colon \mathbb{R}^d \to \mathbb{R}$ be a sequence of continuous functions with $\operatorname{spt}\varphi_i \subset B(0,1/i)$ and $\int \varphi_i \, dx \equiv 1$. Let μ be a locally finite Borel measure on $\mathbb{R}^d$, and recall that $\varphi_i * \mu$ is the function

$$\varphi_i * \mu(x) = \int \varphi_i(x-y)\, d\mu(y).$$

Check that $\varphi_i * \mu$ is continuous. Then, note that $\varphi_i * \mu$ can also be viewed as a Borel measure, if we write $[\varphi_i * \mu](B) = \int_B \varphi_i * \mu \, dx$. Show that $\varphi_i * \mu \rightharpoonup \mu$ as $i \to \infty$. Thus, any locally finite Borel measure can be approximated weakly by continuous functions.

Exercise 2 (1 point)**.** This exercise asks you to spend some time on the proof of Frostman's lemma (Lemma 3.8). Note that $E = [1/2, 1/2]^2 \subset [0,1)^2$ is a compact set satisfying

$$\mathcal{H}^1(E) = \infty > 0.$$

So, the proof of Frostman's lemma with exponent $s = 1$ gives you a non-trivial 1-Frostman measure $\mu \in \mathcal{M}(E)$. Which measure is it? (If there are some annoying constants, just ignore them, and concentrate on describing the measure qualitatively).

Exercise 3 (2 points)**.** Prove inequality (3.33) from the lecture notes in the following special case: if $A, B \subset \mathbb{R}$ are compact, then

$$\dim_{\mathrm{H}}(A \times B) \leq \dim_{\mathrm{H}} A + \overline{\dim}_{\mathrm{B}} B.$$

Exercise 4 (2 points)**.** This exercise asks you to spend some time on the proof of Marstrand's projection theorem (Theorem 4.7) by showing the following sharpened version (due to R. Kaufman): if $E \subset \mathbb{R}^2$ is Borel with $\dim_{\mathrm{H}} E =: s \in [0,1]$, then

$$\dim_{\mathrm{H}}\{e \in S^1 : \dim_{\mathrm{H}} \pi_e(E) < s\} \leq s.$$

Hints: Pick $t < s$ and a measure $\mu \in \mathcal{M}(E)$ with $I_t(\mu) < \infty$.

Make a counter assumption that $\dim_{\mathrm{H}}\{e \in S^1 : \dim_{\mathrm{H}} \pi_e(E) < t\} > t$. Wait a second: why is this even a counter assumption? Find an t-Frostman measure $\sigma \in \mathcal{M}(\{e \in S^1 : \dim_{\mathrm{H}} \pi_e(E) < t\})$. You may take for granted that $\{e \in S^1 : \dim_{\mathrm{H}} \pi_e(E) < t\}$ is Borel. Then, integrate $I_t(\pi_e \mu)$ with respect to σ and see what happens. Why do you reach a contradiction?

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