Introduction to Quantum Computation
Master's Programme in Mathematics and Statistics
Fall 2018
Exercise set 4

Exercise 1. Assume that $|\psi\rangle$ is an entangled two-qubit state. Show that for any unitary one-qubit transformations U and V, also the state $(U \otimes V)\,|\psi\rangle$ is entangled.

Exercise 2. Assume $|\psi\rangle = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is a two-qubit state with $a, b, c, d \in \mathbb{C}$. Show that $|\psi\rangle$ is entangled if and only if $ad \neq bc$. (Hint: see next page.)

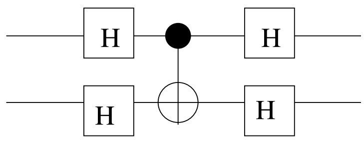
Exercise 3. Prove the no-cloning theorem: There is no two-qubit unitary operator U and state $|s\rangle$ such that for any state $|\psi\rangle$

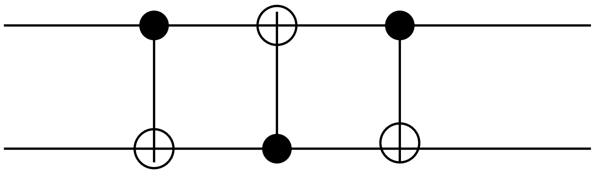
$$U(|\psi\rangle \otimes |s\rangle) = |\psi\rangle \otimes |\psi\rangle\,.$$

(Hint: see next page.)

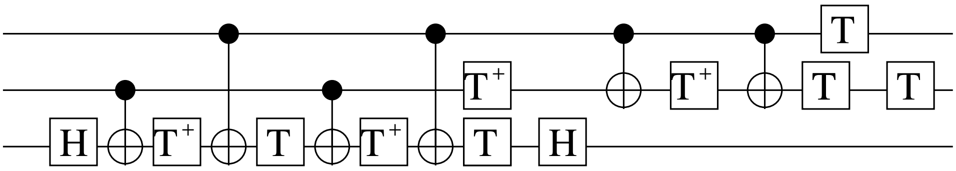
Exercise 4. What is the matrix representation of the Toffoli gate?

Exercise 5. What do the following circuits do? Express the corresponding matrices.

(a)


(b)


Exercise 6. In what sense does the following circuit implement a Toffoli gate? (H is the Hadamard gate, T is the gate corresponding to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$.)



Hint for exercise 2: For the harder direction, consider what happens if one applies an operator of the form $r\begin{bmatrix} c & -a \\ a^* & c^* \end{bmatrix}$ *to the first qubit.*

Hint for exercise 3: Show that any two states cloned are either the same or orthogonal.